Final Examination Example

Seven questions × 4 points = 28 points

Remember to explain your approach.

1) (6.1.4) Find $F(s) = \mathcal{L}[f(t)]$ for
$f(t) = \cos^2 \omega t$

Solution: The Laplace transform is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

From trig. identity $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \Big| \Rightarrow$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Laplace transform is linear $\forall a, b \in \mathbb{R}$

$$\mathcal{L}[af + bg] = a\mathcal{L}(f) + b\mathcal{L}(g)$$

$$\mathcal{L}[\cos^2\omega t] = \frac{1}{2}\mathcal{L}[1] + \frac{1}{2}\mathcal{L}[\cos(2\omega t)]$$

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}\cos(2\omega t) = \frac{s}{s^2 + (2\omega)^2}$$

$$\mathcal{L}[\cos^2\omega t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 4\omega^2}\right].$$

2) (6.1.26) Find $f(t) = \mathcal{L}^{-1}[F(s)]$ for
$F(s) = \dfrac{5s+1}{s^2 - 25}$

Solution: Use $\mathcal{L}[\sinh at] = \dfrac{a}{s^2 - a^2}$

$$\mathcal{L}[\cosh at] = \frac{s}{s^2 - a^2}$$

$$F(s) = 5\frac{s}{s^2 - 5^2} + \frac{1}{5}\frac{5}{s^2 - 5^2}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = 5\cosh 5t + \frac{1}{5}\sinh 5t$$
$$= \frac{13}{5}e^{5t} + \frac{12}{5}e^{-5t}.$$

3) (6.2.5) Solve the IVP
$y'' - \frac{1}{4}y = 0$, $y(0) = 12$, $y'(0) = 0$ by Laplace transform

Solution: Take Laplace transform using

$$\mathcal{L}[y'] = sY(s) - y(0)$$
$$\mathcal{L}[y''] = s^2 Y(s) - sy(0) - y'(0)$$

$$\left(s^2 - \frac{1}{4}\right)Y(s) - 12s = 0 \Rightarrow$$

$$Y(s) = \frac{12s}{s^2 - \frac{1}{4}} \Rightarrow$$

$$y(t) = 12\cosh\frac{t}{2}.$$

4) (6.5.8) Solve the integral $\mathcal{L}$.

$$(1) \quad y(t) + 4\int_0^t y(\tau)(t - \tau)\,d\tau = 2t$$

by Laplace transforms.

Solution: The Laplace transform of a convolution is

$$\mathcal{L}[f*g] = \mathcal{L}(f)\cdot\mathcal{L}(g) = F(s)\,G(s)$$

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

Apply to (1) & obtain:

$$Y(s) + 4\,Y(s)\,\frac{1}{s^2} = \frac{2}{s^2} \Rightarrow$$

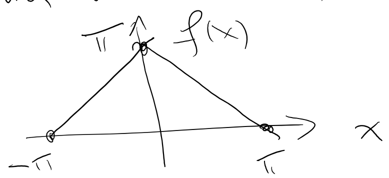
using $\mathcal{L}[t] = \frac{1}{s^2}$

$$Y(s) = \frac{2}{s^2+4} \Rightarrow$$

$$y(t) = \cos(2t).$$

5) Find the Fourier series of $f(x)$,

$f(x+2\pi) = f(x)$



Solution: The Fourier series is

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right)$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{\pi}{2}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos nt\,dt$$

$$= 2\int_0^{\pi}\cos nt\,dt - \frac{2}{\pi}\int_0^{\pi} t\cos nt\,dt$$

$$\int_0^{\pi}\cos nt\,dt = \frac{1}{n}\sin nt\Big|_0^{\pi} = 0$$

$$\int_0^{\pi} t\cos nt\,dt = \frac{t}{n}\sin nt\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nt\,dt =$$

Int. by parts $dv = \cos nt\,dt \Rightarrow v = \frac{1}{n}\sin nt$

$u = t \Rightarrow du = dt$

$$= \frac{1}{n^2} \cos(nt)\Big|_0^{\pi} = \frac{(-1)^n - 1}{n^2}$$

$$a_n = -\frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] = \frac{2\left[1-(-1)^n\right]}{\pi n^2}$$

n even $a_{2k} = 0$

n odd $a_{2k+1} = \dfrac{4}{\pi(2k+1)^2}$

$b_n = 0$ since $f(t) = f(-t)$ even

$$f(t) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos[(2k+1)t]}{(2k+1)^2} .$$

6) (11.4.12) Using the Parseval identity

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx$$

and $f(x) = x^2$, for $x \in [-\pi, \pi]$,

$f(x) = f(x + 2\pi)$ prove that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Solution: $f(x)$ is even $f(x) = f(-x)$

hence has Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{3\pi}\, x^3\Big|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

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$$a_0 = \frac{1}{2\pi}\int_{-\pi} f(x)\,dx = \frac{2}{\pi}\int_{-\pi} x^2\,dx = \frac{}{3\pi}\, x \Big|_{-\pi} = \frac{}{3}$$

$n > 1$ $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$$

Integrate by parts twice:

$$a_n = \frac{1}{\pi}\left[\frac{x^2\sin(nx)}{n}\Big|_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x\sin(nx)\,dx \right]$$

$$= \frac{2}{\pi n}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= \frac{2}{\pi n}\left[-x\frac{\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx \right]$$

$$= -\frac{2}{\pi n^2}\left[\pi\cos(n\pi) + \pi\cos(-n\pi) \right]$$

$$= 4\frac{(-1)^n}{n^2}$$

Obtain

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$$

Compute

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx =$$

$$\frac{1}{\pi}\int_{-\pi} f(x) \quad \pi \quad {}_{-\pi}$$

$$= \frac{2\pi^4}{5}$$

$$\frac{\pi^4}{15}$$

Parseval's identity states

$$2\left(\frac{\pi^2}{3}\right)^2 + 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5} \Rightarrow$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{2\pi^4}{5} - \frac{2\pi^4}{9}\right)\frac{1}{16}$$

$$= \frac{4\pi^4}{45}\,\frac{1}{8} = \frac{\pi^4}{90} \quad \checkmark$$

7) Consider the Fourier expansion of

$f(x) = \cos\frac{x}{2\pi}$ using the periodic

basis functions $B_0(x), B_1(x), B_2(x)$

$B_n(x) = B_n(x+1)$

on $x \in (0,1]$, $B_n(x) = \text{sign}\left(\cos\frac{nx}{2\pi}\right)$

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases}$$ stated as

$-1 \quad x < -1;$

$$f(x) = \sum_{n=0}^{\infty} a_n B_n(x).$$

Describe the expected behavior of a_n with increasing n.

Solution: The Basis set has discontinuities at $x_k = \pi^2 \frac{2k+1}{n}$ $\quad k = 0, 1, \ldots$

$$\cos\left(\frac{n x_k}{2\pi}\right) = \cos\left[\frac{\pi}{2}(2k+1)\right] = 0$$

while $f(x) \in C^{\infty}$ is continuous with continuous derivatives, and is hence non-conformant with respect to the basis set. Gibbs phenomenon will occur and

$$a_n \sim O\left(\frac{1}{n}\right)$$

Slow decay of Fourier coefficients.