Homework 01 Solution

Follow this model when solving your homework.

8 exercises x .25 = 2 points, 4 problems x .5 = 2 points, 1 project x 1 points = 1 point

1 Exercises

Exercise 1. PS1.1.4

SOLUTION. $y' = -1.5y$ is: 1st-order, explicit, linear, separable ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1.5y \Rightarrow \frac{\mathrm{d}y}{y} = -1.5\mathrm{d}x \Rightarrow \int \frac{\mathrm{d}y}{y} = -1.5\int \mathrm{d}x + \ln\mathrm{A} \Rightarrow \ln y = -1.5x + \ln A \Rightarrow y(x) = Ae^{-1.5x}$$

```
In[19]:= ODE = y'[x]==1.5y[x]
```

$y'(x) = 1.5\,y(x)$

```
In[20]:= DSolve[ODE,y[x],x]
```

$\{\{y(x) \to c_1\,e^{1.5x}\}\}$

```
In[21]:=
```

Exercise 2. PS1.1.6

SOLUTION. $y'' = -y$ is: 2nd-order, explicit, linear, solve by finding roots of characteristic equation $r^2 + 1 = 0$

$$r_{1,2} = \pm i \Rightarrow y(x) = a\,e^{ix} + b\,e^{-ix} = A\sin(x) + B\cos(x)$$

```
In[21]:= ODE = y''[x]==-y[x]
```

$y''(x) = -y(x)$

```
In[22]:= DSolve[ODE,y[x],x]
```

$\{\{y(x) \to c_2 \sin(x) + c_1 \cos(x)\}\}$

```
In[23]:=
```

Exercise 3. PS1.1.7

SOLUTION. $y' = \cosh(5.13x)$ is: 1st-order, explicit, linear, separable ODE

```
In[23]:= ODE = y'[x]==Cosh[5.13x]
```

$y'(x) = \cosh(5.13\,x)$

```
In[24]:= DSolve[ODE,y[x],x]
```

$\{\{y(x) \to c_1 + 0.194932 \sinh(5.13\,x)\}\}$

```
In[25]:=
```

Exercise 4. PS1.1.8

SOLUTION. $y''' = e^{-0.2x}$ is: 3rd-order, explicit, linear, separable ODE

```
In[25]:= ODE = y'''[x]==Exp[-0.2x]
```

$y^{(3)}(x) = e^{-0.2x}$

```
In[26]:= DSolve[ODE,y[x],x]
```

$\{\{y(x) \to c_3 x^2 + c_2 x + c_1 - 125.\, e^{-0.2x}\}\}$

```
In[27]:=
```

Exercise 5. PS1.3.3

SOLUTION. $y' = \sec^2 y$ is: 1st-order, explicit, non-linear, separable ODE

$$\frac{\mathrm{d}y}{\sec^2 y} = \cos^2 y\,\mathrm{d}y = \mathrm{d}x \Rightarrow \int \cos^2 y\,\mathrm{d}y = x + c$$

From trigonometric identity $\cos(2y) = \cos^2 y - \sin^2 y = 2\cos^2 y - 1 \Rightarrow \cos^2 y = \frac{1}{2}(\cos(2y) + 1)$, so

$$\int \cos^2 y\,\mathrm{d}y = \frac{1}{4}\sin(2y) + \frac{y}{2} = x + c.$$

```
In[63]:= ODE = y'[x]==Sec[y[x]]^2
```

$y'(x) = \sec^2(y(x))$

```
In[64]:= DSolve[ODE,y[x],x]
```

$\left\{\left\{y(x) \to \text{InverseFunction}\left[2\left(\frac{\#1}{2} + \frac{1}{4}\sin(2\#1)\right)\&\right][c_1 + 2x]\right\}\right\}$

```
In[65]:= Integrate[Cos[y]^2,y]
```

$\frac{y}{2} + \frac{1}{4}\sin(2y)$

```
In[66]:=
```

Exercise 6. PS1.3.4

SOLUTION. $y' \sin 2\pi x = \pi y \cos 2\pi x$ is: 1st-order, explicit, non-linear, separable ODE

$$\frac{y'}{y} = \pi\frac{\cos 2\pi x}{\sin 2\pi x} = \pi \cot 2\pi x \Rightarrow \frac{\mathrm{d}y}{y} = \pi \cot 2\pi x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}(\log(\sin 2\pi x)) \Rightarrow \log y = \frac{1}{2}\log(\sin 2\pi x) + \log c \Rightarrow$$

$$y(x) = c\sqrt{\sin 2\pi x}.$$

```
In[67]:= ODE = y'[x]==Pi y[x] Cos[2Pi x]/Sin[2 Pi x]
```

$y'(x) = \pi\, y(x) \cot(2\pi x)$

```
In[68]:= DSolve[ODE,y[x],x]
```

$\{\{y(x) \to c_1 \sqrt{\sin(2\pi x)}\}\}$

Exercise 7. PS1.3.5

SOLUTION. $yy' + 36x = 0$ is: 1st-order, explicit, non-linear, separable ODE

$$\frac{1}{2}\mathrm{d}\,y^2 = -36\,x\,\mathrm{d}x \Rightarrow \frac{y^2}{2} = -18x^2 + c$$

```
In[69]:= ODE = y[x] y'[x]+36 x ==0
```

$y(x)\, y'(x) + 36\, x = 0$

```
In[70]:= DSolve[ODE,y[x],x]
```

$\left\{\{y(x) \to -\sqrt{2}\sqrt{c_1 - 18x^2}\}, \{y(x) \to \sqrt{2}\sqrt{c_1 - 18x^2}\}\right\}$

```
In[71]:=
```

Exercise 8. PS1.3.5

SOLUTION. $yy' + 36x = 0$ is: 1st-order, explicit, non-linear, separable ODE

$$\frac{1}{2}\mathrm{d}\,y^2 = -36x\,\mathrm{d}x \Rightarrow \frac{y^2}{2} = -18x^2 + c$$

```
In[69]:= ODE = y[x] y'[x]+36 x ==0
```

$y(x)\,y'(x) + 36\,x = 0$

```
In[70]:= DSolve[ODE,y[x],x]
```

$\left\{\left\{y(x) \to -\sqrt{2}\,\sqrt{c_1 - 18\,x^2}\right\}, \left\{y(x) \to \sqrt{2}\,\sqrt{c_1 - 18\,x^2}\right\}\right\}$

```
In[71]:=
```

2 Problems

Problem 1. PS1.2.2

SOLUTION. $F(x, y, y') = yy' + 4x = 0$, is 1st-order, non-linear, separable, trivially implicit since $F = 0$ can easily be solved to find $y' = f(x, y) = -4x/y$. Note that $f(x, y)$ is not Lipschitz continuous, hence existence of a unique solution is not guaranteed.

```
In[15]:= f[x_,y_]=-4x/y
```

$-\frac{4\,x}{y}$

```
In[16]:= ODE = y'[x]==f[x,y[x]]
```

$y'(x) = -\frac{4\,x}{y(x)}$

```
In[17]:= DSolve[ODE,y[x],x]
```

$\left\{\left\{y(x) \to -\sqrt{2}\,\sqrt{c_1 - 2\,x^2}\right\}, \left\{y(x) \to \sqrt{2}\,\sqrt{c_1 - 2\,x^2}\right\}\right\}$

```
In[18]:=
```

From Slide8Lesson01

```
In[18]:= Off[DSolve::bvnul];
         sol1[x_] = y[x] /. DSolve[{ODE,y[1]==1},y[x],x][[1,1]];
         sol2[x_] = y[x] /. DSolve[{ODE,y[0]==2},y[x],x][[1,1]];
         {sol1[x],sol2[x]}
```

$\left\{\sqrt{5 - 4\,x^2}, 2\,\sqrt{1 - x^2}\right\}$

```
In[19]:=
```

From Slide8Lesson01:

```
In[20]:= DirectionField = VectorPlot[{1,f[x,y]},{x,0,1.25},{y,0,2.5},Axes->True,Frame-
         >False,ImageSize->Large,VectorScale->.3,VectorMarkers->None];
         SolutionPlot = Plot[{sol1[x],sol2[x]},{x,0,1.25},ImageSize->Large];
         plots=Show[{SolutionPlot,DirectionField}];
         Export["/home/student/courses/MATH528/HW01Fig01.pdf",plots]
```

/home/student/courses/MATH528/HW01Fig01.pdf

```
In[21]:=
```

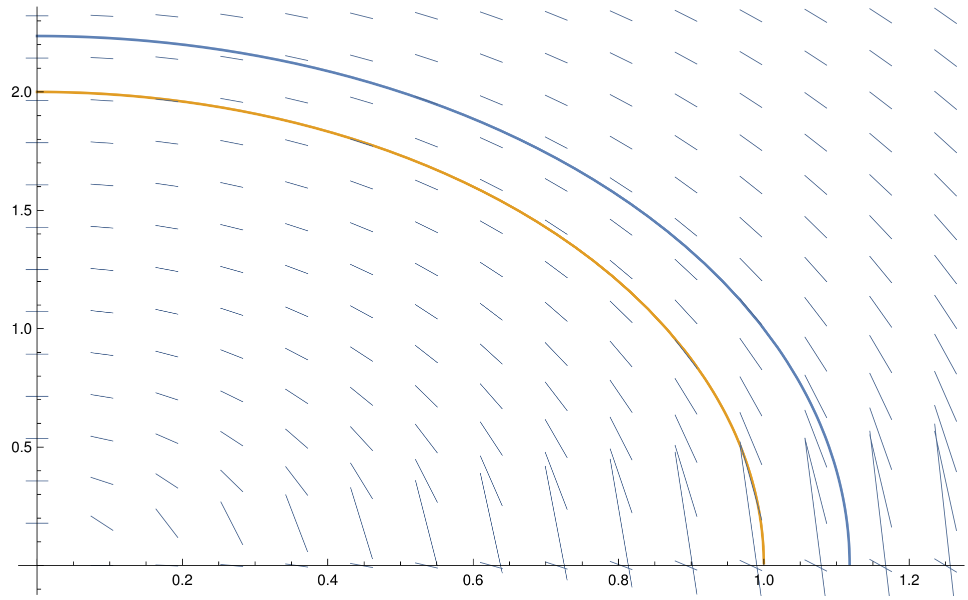

Figure 1. PS1.2.2 direction field and solutions passing through $(1,1),(0,2)$

Problem 2. PS1.2.3

SOLUTION. $y' = f(y) = 1 - y^2$ is: 1st-order, explicit, non-linear

```
In[21]:= f[x_,y_]=1-y^2;
         ODE = y'[x]==f[x,y[x]]
```

$$y'(x) = 1 - y(x)^2$$

```
In[22]:= DSolve[ODE,y[x],x]
```

$$\left\{\left\{y(x) \to \frac{e^{2x} - e^{2c_1}}{e^{2c_1} + e^{2x}}\right\}\right\}$$

```
In[23]:= Off[Solve::ifun];
         sol1[x_] = y[x] /. DSolve[{ODE,y[0]==0},y[x],x][[1,1]];
         sol2[x_] = y[x] /. DSolve[{ODE,y[2]==1/2},y[x],x][[1,1]];
         {sol1[x],sol2[x]}
```

$$\left\{\frac{e^{2x} - 1}{e^{2x} + 1}, \frac{3\,e^{2x} - e^4}{3\,e^{2x} + e^4}\right\}$$

```
In[24]:=
```

Direction field and solutions

```
In[24]:= DirectionField = VectorPlot[{1,f[x,y]},{x,0,2},{y,-1,1},Axes->True,Frame->False,
         ImageSize->Large,VectorScale->.051,VectorMarkers->None,VectorStyle->Pink];
         SolutionPlot = Plot[{sol1[x],sol2[x]},{x,0,2},ImageSize->Large];
         plots=Show[{SolutionPlot,DirectionField}];
         Export["/home/student/courses/MATH528/HW01Fig02.pdf",plots]
```

/home/student/courses/MATH528/HW01Fig02.pdf

```
In[25]:=
```

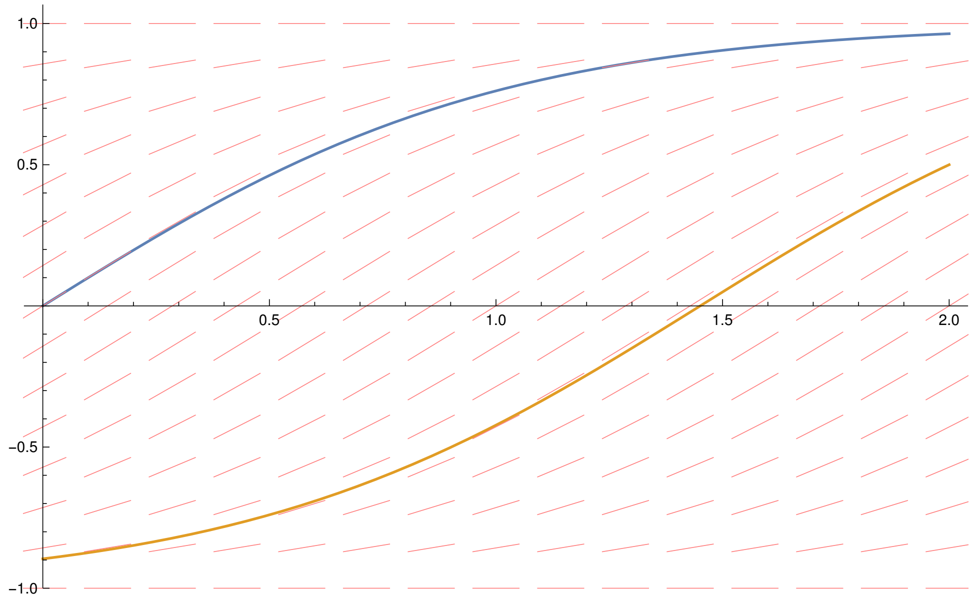

Figure 2. PS1.2.2 direction field and solutions passing through $(1,1),(0,2)$

Problem 3. PS1.1.16

SOLUTION. $y'^2 - xy' + y = 0$ is 1st-order, non-linear, implicit. Verify solutions $y(x) = c(x-c)$, $y(x) = x^2/4$

```
In[1]:= ODE = y'[x]^2 - x y'[x] + y[x] == 0
```

$y'(x)^2 - x\,y'(x) + y(x) = 0$

```
In[5]:= sol1[x_]=c(x-c)
```

$c\,(x-c)$

```
In[6]:= Simplify[ODE /. y->sol1]
```

True

```
In[7]:= sol2[x_]=x^2/4
```

$\frac{x^2}{4}$

```
In[8]:= Simplify[ODE /. y->sol2]
```

True

```
In[9]:= DSolve[ODE,y[x],x]
```

$\{\{y(x) \to c_1\,x - c_1^2\}\}$

```
In[10]:= DSolve[{ODE,y[0]==0},y[x],x]
```

$\{\{y(x) \to 0\}\}$

```
In[11]:=
```

Singular solution $y(x) = x^2/4$ is the envelope of all particular solutions.

Problem 4. PS1.1.18

SOLUTION. Decay follows law $y' = -ky$ with solution $y(t) = e^{-kt}y(0)$, The half-life τ satisfies equation

$$\frac{y(\tau)}{y(0)} = \frac{1}{2} = e^{-k\tau} \Rightarrow -k\tau = -\log 2 \Rightarrow k = \frac{\log 2}{3.6}\left[\frac{1}{\text{day}}\right]$$

After 1 day $y(1) = \exp\left[-\frac{\log 2}{3.6} \times 1\right] \times 1\,[\text{gram}] = .825\,[\text{gram}]$. After 1 year $y \sim 3 \times 10^{-31}\,[\text{gram}]$, undetectable since a single atom would weigh $224\,\text{amu} = 224 \times 1.66 \times 10^{-24}$ gram

```
In[11]:= k=Log[2.]/3.6; y[t_,y0_]=Exp[-k t] y0
```

$e^{-0.192541t}\text{y0}$

```
In[12]:= y[1.,1.]
```
0.824861
```
In[13]:= y[365.25,1.]
```
2.870769718761845`*^-31
```
In[14]:= 224 1.66 10^(-24)
```
3.7184`*^-22
```
In[15]:=
```

3 Projects

3.1 PS2.1.16

(a) The larger field suggests a pole in quadrant II

```
In[28]:= f[x_,y_] = x + y;
         DirectionField = VectorPlot[{1,f[x,y]},{x,-5,2},{y,-1,5},Axes->True,Frame->False,
         ImageSize->Large,VectorScale->.051,VectorMarkers->None,VectorStyle->Pink];
         Export["/home/student/courses/MATH528/HW01Fig03.pdf",DirectionField]
```
/home/student/courses/MATH528/HW01Fig03.pdf
```
In[29]:=
```

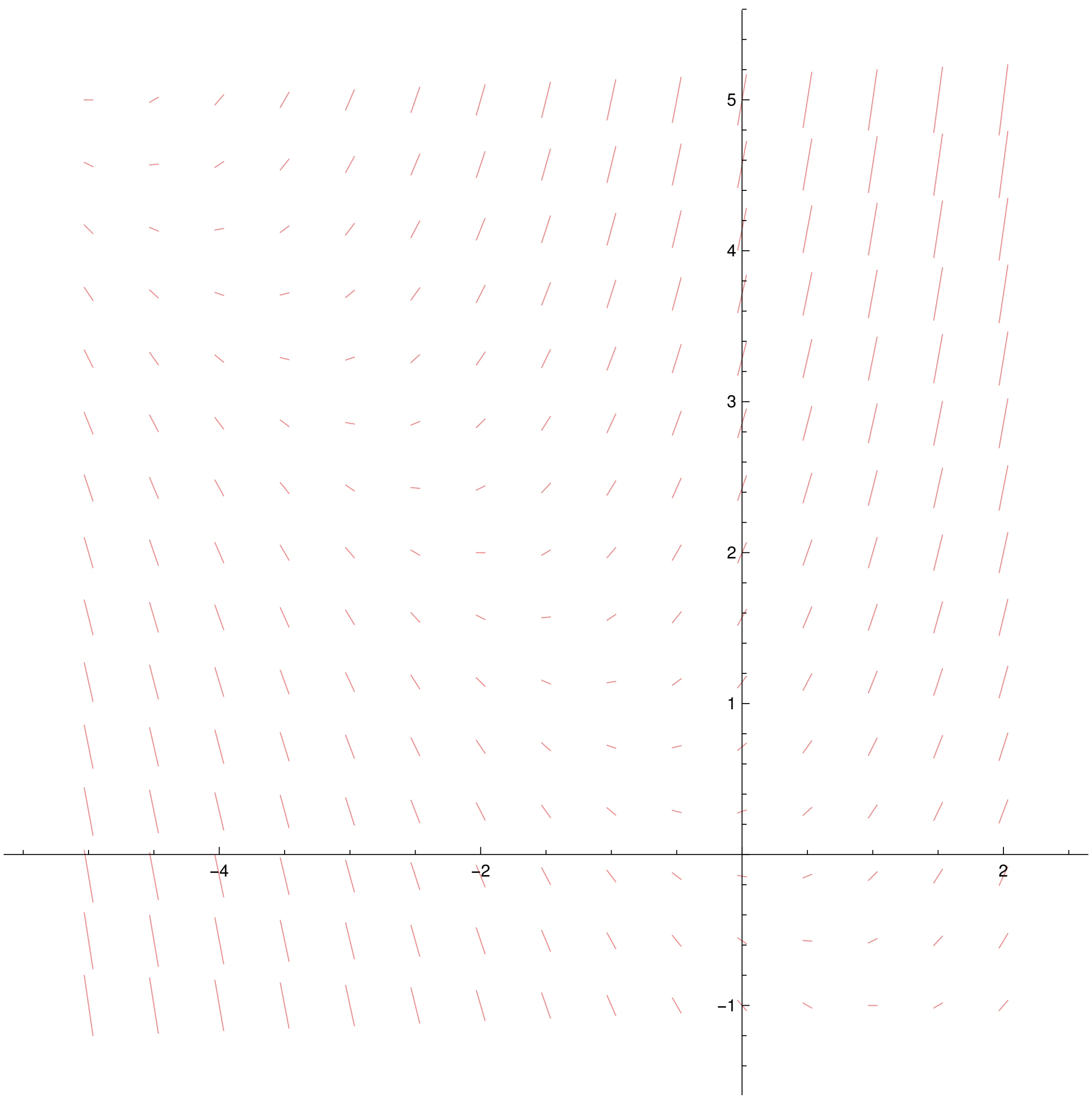

Figure 3. Direction field of ODE $y' = x + y$

(b) From $x^2+9y^2(x)=c$ differentiation by x gives $x+9yy'=0 \Rightarrow y'=-x/(9y)$

```
In[35]:= f[x_,y_] = -x/(9y);
         DirectionField = VectorPlot[{1,f[x,y]},{x,-3,3},{y,0,3},Axes->True,Frame->False,
         VectorMarkers->None,VectorScale->.051,VectorStyle->Pink,AspectRatio->Automatic];
         Export["/home/student/courses/MATH528/HW01Fig04.pdf",DirectionField]
```

/home/student/courses/MATH528/HW01Fig04.pdf

```
In[36]:=
```

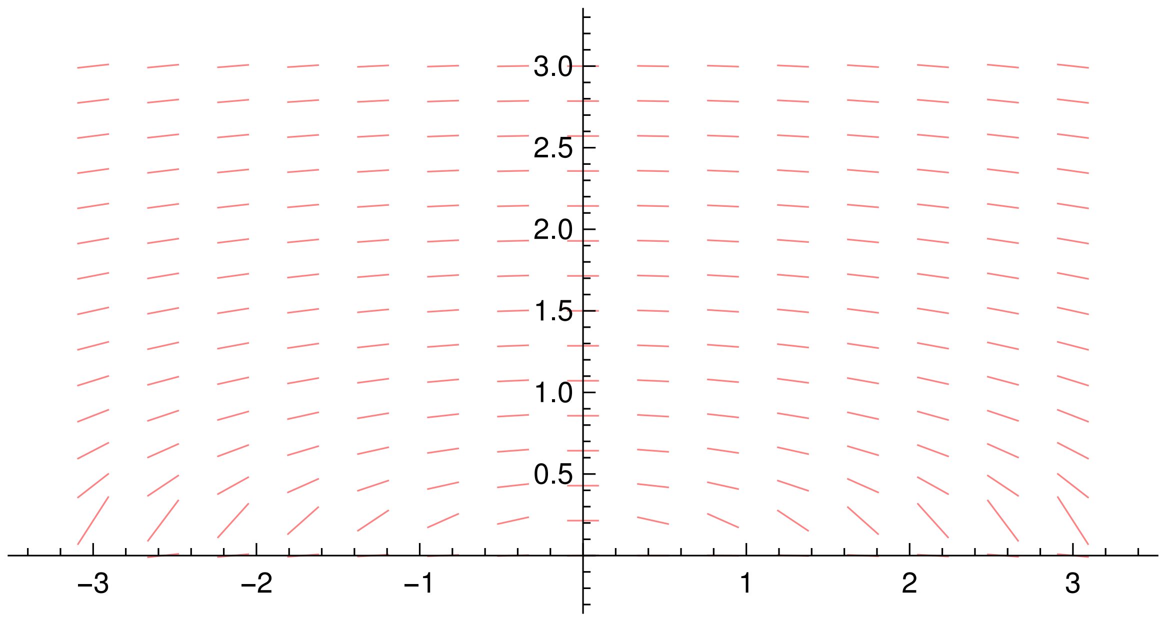

Figure 4. Direction field of ODE $y'=-x/(9y)$

Figure	Implicit equation	ODE
circles	$x^2+y^2=c$	$x+yy'=0$
hyperbolas	$xy=c$	$y+xy'=0$
parabolas	$y=x^2$+c	$y'=2x$

Table 1. ODEs corresponding to conics

(c) From Table 1, solutions to $y'=-x/y$ are circles. Confirm by drawing direction field

```
In[38]:= f[x_,y_] = -x/y;
        DirectionField = VectorPlot[{1,f[x,y]},{x,-2,2},{y,0,2},Axes->True,Frame->False,
        VectorMarkers->None,VectorScale->.1,VectorStyle->Pink,AspectRatio->Automatic];
        Export["/home/student/courses/MATH528/HW01Fig05.pdf",DirectionField]
```

/home/student/courses/MATH528/HW01Fig05.pdf

```
In[39]:=
```

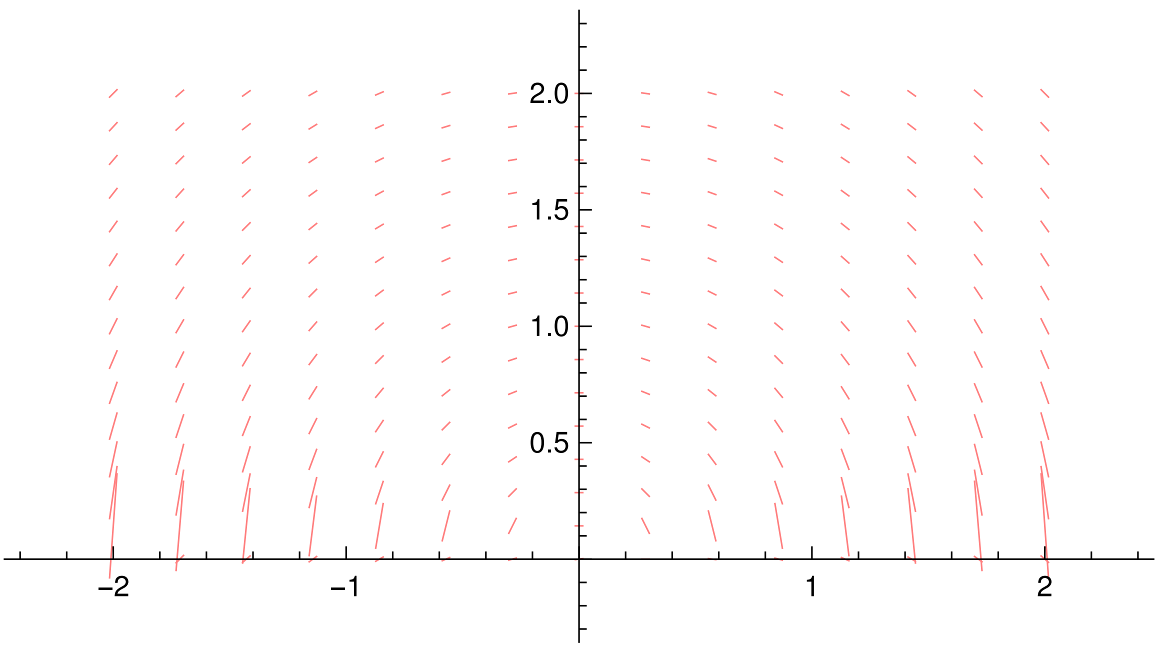

Figure 5. Direction field of ODE $y'=-x/(9y)$

(d) Solutions to $y'=-y/2$ are exponentials. Confirm by drawing direction field

```
In[39]:= f[x_,y_] = -y/2;
        DirectionField = VectorPlot[{1,f[x,y]},{x,-2,2},{y,-2,2},Axes->True,Frame->False,
        VectorMarkers->None,VectorScale->.1,VectorStyle->Pink,AspectRatio->Automatic];
        Export["/home/student/courses/MATH528/HW01Fig06.pdf",DirectionField]
```

/home/student/courses/MATH528/HW01Fig06.pdf

```
In[40]:=
```

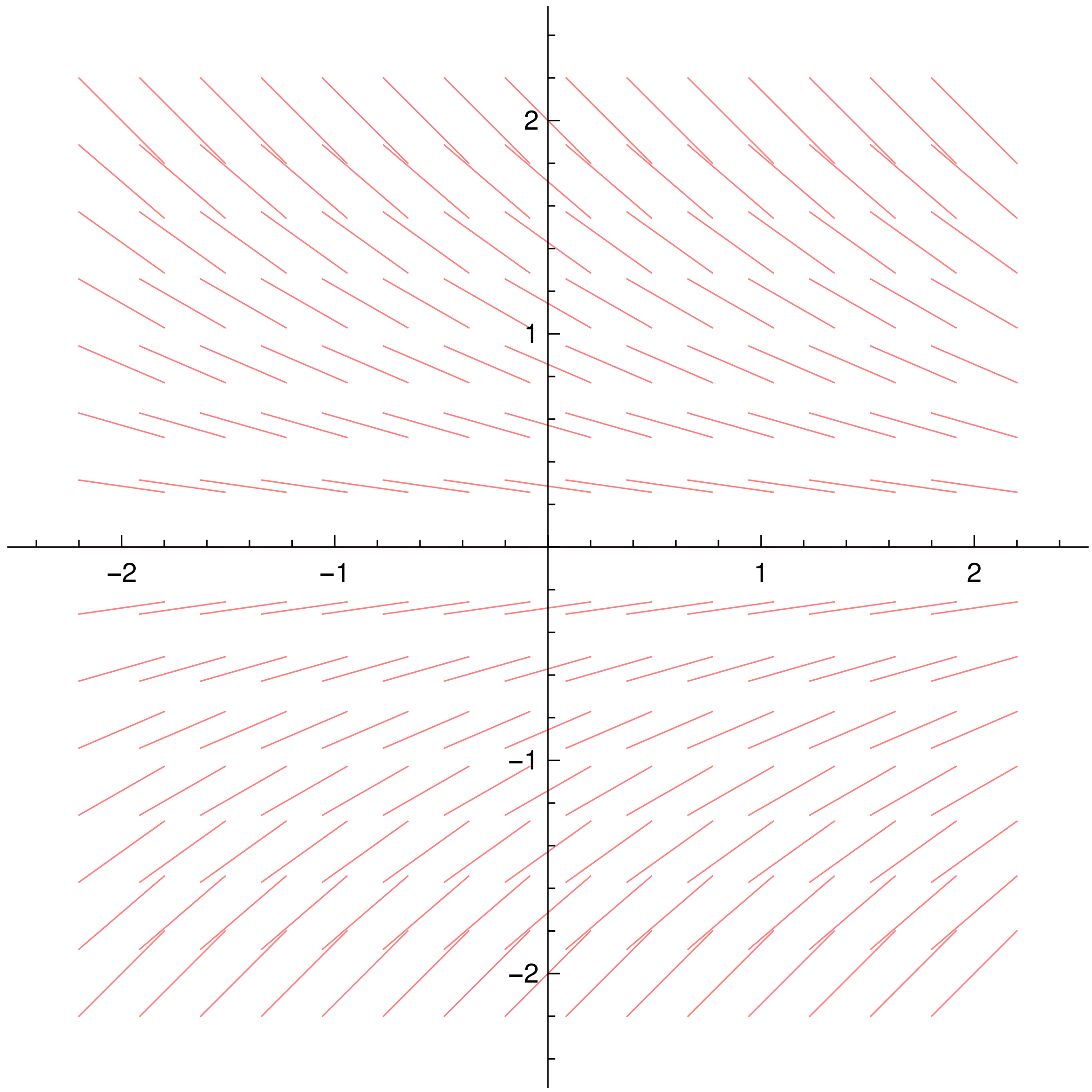

Figure 6. Direction field of ODE $y' = -x/(9y)$