Homework 02 Solution

1 Exercises

Exercise. PS1.4.1

Solution. $2xy\,\mathrm{d}x + x^2\,\mathrm{d}y$ is an exact differential

```
In[1]:= df = 2 x y dx + x^2 dy
```

$2\,\mathrm{d}x x\,y + \mathrm{d}y x^2$

```
In[4]:= {P[x_,y_],Q[x_,y_]} = {Coefficient[df,dx,1],Coefficient[df,dy,1]}
```

$\{2\,x\,y, x^2\}$

```
In[5]:= ExactDF = D[P[x,y],y] == D[Q[x,y],x]
```

True

```
In[6]:=
```

Exercise. PS1.4.2

Solution. $x^3\,\mathrm{d}x + y^3\,\mathrm{d}y$ is an exact differential

```
In[6]:= df = x^3 dx + y^3 dy;
        {P[x_,y_],Q[x_,y_]} = {Coefficient[df,dx,1],Coefficient[df,dy,1]};
        ExactDF = D[P[x,y],y] == D[Q[x,y],x];
        {P[x,y],Q[x,y],ExactDF}
```

$\{x^3, y^3, \text{True}\}$

```
In[7]:=
```

Exercise. PS1.4.3

Solution. $\sin x \cos y\,\mathrm{d}x + \cos x \sin y\,\mathrm{d}y$ is an exact differential

```
In[7]:= df = Sin[x] Cos[y] dx + Cos[x] Sin[y] dy;
        {P[x_,y_],Q[x_,y_]} = {Coefficient[df,dx,1],Coefficient[df,dy,1]};
        ExactDF = D[P[x,y],y] == D[Q[x,y],x];
        {P[x,y],Q[x,y],ExactDF}
```

$\{\sin(x)\cos(y), \cos(x)\sin(y), \text{True}\}$

```
In[8]:=
```

Exercise. PS1.4.4

Solution. $e^{3\theta}(\mathrm{d}r + 3r\,\mathrm{d}\theta)$ is an exact differential

```
In[8]:= df = Exp[3y](dx + 3x dy);
        {P[x_,y_],Q[x_,y_]} = {Coefficient[df,dx,1],Coefficient[df,dy,1]};
        ExactDF = D[P[x,y],y] == D[Q[x,y],x];
        {P[x,y],Q[x,y],ExactDF} /. {x->r, y->theta}
```

$\{e^{3\theta}, 3\,r\,e^{3\theta}, \text{True}\}$

```
In[9]:=
```

Exercise. PS1.5.3

Solution. $y' - y = 5.2$ is a linear, first-order ODE of form $y' + p(x)\,y = r(x)$, with $p(x) = -1$, $r(x) = 5.2$, and general solution

$$y(x) = e^{-h(x)}\left(\int e^{h(x)} r(x)\,\mathrm{d}x + c\right), h = \int p(x)\,\mathrm{d}x,$$

which gives

$$h(x) = -x$$

$$y(x) = e^{x}\left(5.2\int e^{-x}\,\mathrm{d}x + c\right) = -5.2 + c e^{x}.$$

Verify:

```
In[23]:= ODE = y'[x] - y[x] == 5.2;
         sol[t] = y[x] /. DSolve[ODE,y[x],x][[1,1]]
```

$c_1\, e^{1.x} - 5.2$

```
In[25]:= Integrate[-4 x Exp[-2x],x]
```

$-4\, e^{-2x}\left(-\frac{x}{2} - \frac{1}{4}\right)$

```
In[26]:=
```

Exercise. PS1.5.4

SOLUTION. $y' = 2y - 4x$, linear ODE with $p(x) = -2$, $r(x) = -4x$, $h(x) = -2x$

$$y(x) = e^{2x}\left(-4\int x e^{-2x}\,\mathrm{d}x + c\right) = e^{2x}\left(2x e^{-2x} - \frac{1}{2}\int e^{-2x}\,\mathrm{d}x + c\right) = e^{2x}(2x e^{-2x} + e^{-2x} + c),$$

by integration by parts. Verify:

```
In[24]:= ODE = y'[x] == 2y[x] -4x;
         sol[t] = y[x] /. DSolve[ODE,y[x],x][[1,1]]
```

$c_1\, e^{2x} - 4\left(-\frac{x}{2} - \frac{1}{4}\right)$

```
In[25]:=
```

Exercise. PS1.5.5

SOLUTION. $y' + ky = e^{-kx}$, linear ODE with $p(x) = k$, $r(x) = e^{-kx}$

```
In[27]:= ODE = y'[x] + k y[x] == Exp[-k x];
         sol[t] = y[x] /. DSolve[ODE,y[x],x][[1,1]]
```

$c_1\, e^{-kx} + x\, e^{-kx}$

```
In[28]:=
```

Exercise. PS1.5.6

SOLUTION. $y' + 2y = 4\cos 2x$, $y(\pi/4) = 3$, is a linear ODE with $p(x) = 2$, $r(x) = 4\cos 2x$

```
In[29]:= ODE = y'[x] + 2 y[x] == 4 Cos[2x];
         iCond = y[Pi/4]==3;
         sol[t] = Expand[y[x] /. DSolve[{ODE,iCond},y[x],x][[1,1]]]
```

$2\, e^{\frac{\pi}{2} - 2x} + \sin(2\,x) + \cos(2\,x)$

```
In[30]:=
```

2 Problems

Problem. PS1.5.31 Newton's law of cooling

SOLUTION. Assume cooling rate is proportional to temperature difference $T(t) - T_r$, with time denoted by t and T_r the room temperature. The problem can be formulated as

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{d}t} = -r\,(T - T_r) \\ T(0) = 300 \end{cases}$$

and also written

$$\begin{cases} \frac{\mathrm{d}Y}{\mathrm{d}t} = -rY \\ Y(0) = 240 \end{cases}$$

with $Y = T(t) - T_r$, with solution $Y(t) = 240e^{-rt}$, or $T(t) = 240e^{-rt} + 60$

```
In[20]:= ODE = T'[t] == -r (T[t]-60);
         sol[t_] = Expand[T[t] /. DSolve[{ODE,T[0]==300},T[t],t][[1,1]]]
```

$240\,e^{-rt} + 60$

```
In[21]:=
```

Find the decay rate r by imposing the condition $T(10) = 200$

```
In[21]:= rsol = FindRoot[sol[10.] == 200., {r,0.1}][[1]]
```

$r \to 0.0538997$

```
In[22]:=
```

Find time at which cake temperature reaches 61 degrees

```
In[22]:= tsol = FindRoot[sol[t]==61 /. rsol,{t,20}][[1]]
```

$t \to 101.682$

```
In[23]:=
```

Verify the solution

```
In[14]:= sol[t] /. {rsol,tsol}
```

61.

Problem. PS1.5.32 Heating and cooling of a building

Solution. Define the ODE and solve

```
In[1]:= Ta[t_] = A - c Cos[2 Pi t/24]
```

$A - c\cos\left(\frac{\pi t}{12}\right)$

```
In[2]:= ODE = T'[t] == Subscript[k,1] (T[t]-Ta[t]) + Subscript[k,2] (T[t] - Subscript[T,w])
        + P
```

$T'(t) = k_1\left(-A + c\cos\left(\frac{\pi t}{12}\right) + T(t)\right) + k_2\,(T(t) - T_w) + P$

```
In[3]:=
```

The above is a linear ODE of form $T' + p(t)\,T = r(t)$, with

$$p(t) = -(k_1 + k_2), r(t) = k_1\left(-A + c\cos\left(\frac{\pi t}{12}\right)\right) - k_2 T_\omega + P$$

The general solution is

$$T(t) = e^{-h(t)}\left(\int e^{h(t)} r(t)\,\mathrm{d}t + B\right), h = \int p(t)\,\mathrm{d}t = -(k_1 + k_2)t.$$

```
In[3]:= h[t_]=-(Subscript[k,1]+Subscript[k,2])t
```

$(k_1 + k_2)(-t)$

```
In[4]:= r[t_]=Subscript[k,1]Ta[t]-Subscript[k,2] Subscript[T,w]+P
```

$k_1\left(A - c\cos\left(\frac{\pi t}{12}\right)\right) - k_2\,T_w + P$

```
In[5]:= T[t_] = FullSimplify[Exp[-h[t]](Integrate[Exp[h[t]] r[t],t]+B)]
```

$$-\frac{A\,k_1 - k_2\,T_w + P}{k_1 + k_2} + B\,e^{(k_1+k_2)t} - \frac{12\,\pi\,c\,k_1\sin\left(\frac{\pi t}{12}\right)}{144\,(k_1 + k_2)^2 + \pi^2} + \frac{144\,c\,k_1\,(k_1 + k_2)\cos\left(\frac{\pi t}{12}\right)}{144\,(k_1 + k_2)^2 + \pi^2}$$

```
In[6]:=
```

The above terms are, in order:

1. temperature balance between ambient, wanted, occupant temperatures
2. decay of initial temperature
3. 4. variation due to change in ambient temperature

A more detailed investigation is presented in mLab03.

Problem. PS1.5.33 Drug injection

SOLUTION. Let $d(t)$ be the amount of drug in the bloodstream. The model is written as

$$d' = A - r d$$

with A the injection rate and r the removal rate per unit of drug in bloodstream. This is a linear ODE

```
In[6]:= ODE = d'[t] == A - r d[t];
        sol[t] = d[t] /. DSolve[ODE,d[t],t][[1,1]]
```

$\frac{A}{r} + c_1 e^{-rt}$

```
In[7]:=
```

Problem. PS1.5.35 Lake Erie

SOLUTION. Volume $V = 450\,\mathrm{km}^3$, flow rate $Q = 175\,\mathrm{km}^3/\mathrm{year}$ (in and out). Let $p(t)$ denote pollution concentration, with $p(0) = 0.04$. The inflow pollution is $p_1 = p(0)/4 = 0.01$. The model is

$$p' = \frac{Q}{V}(p_1 - p), p(0) = 0.04.$$

```
In[7]:= V=450; Q=175; p0=0.04; p1=p0/4; ODE = p'[t] == Q/V (p1-p[t]); iCond = p[0]==p0;
        sol[t] = p[t] /. DSolve[{ODE,iCond},p[t],t][[1,1]]
```

$0.01\,e^{-0.388889t}(3. + 1.\,e^{0.388889t})$

```
In[8]:= FindRoot[sol[t]==p0/2,{t,1}]
```

$\{t \to 2.825\}$

```
In[10]:= 2.825*175
```

494.375

```
In[11]:=
```

3 Projects

3.1 PS2.1.16

The sum $y = y_1 + y_2$ is a solution of $y' + p y = r_1 + r_2$.

Proof. Compute $y' + p y = y_1' + y_2' + p(y_1 + y_2) = (y_1' + p y_1) + (y_2' + p y_2) = r_1 + r_2$