Homework 06

1 Exercises

Exercise. PS4.5.4

Solution. System $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$ has critical points at solutions of $\boldsymbol{f}(\boldsymbol{y}) = 0$,

$$\begin{cases} y_1(4 - y_1) = 0 \\ y_2 = 0 \end{cases}, (y_1, y_2) \in \{(0,0), (4,0)\}.$$

The Jacobian obtained by linearization is

$$\mathbf{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} 4 - 2y_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

At critical point:

- $(0,0)$, $\mathbf{A}$ has eigenvalues (4,1), unstable node
- (4,0), $\mathbf{A}$ has eigenvalues (-4,1), saddle node

Exercise. PS4.5.5

Solution. System $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$ has critical points at solutions of $\boldsymbol{f}(\boldsymbol{y}) = 0$,

$$\begin{cases} y_2 = 0 \\ y_1\left(-1 + \frac{1}{2}y_1\right) = 0 \end{cases}, (y_1, y_2) \in \{(0,0), (2,0)\}.$$

The Jacobian obtained by linearization is

$$\mathbf{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} 0 & 1 \\ -1 + y_1 & 0 \end{pmatrix}.$$

At critical point:

- $(0,0)$, $\mathbf{A}$ has eigenvalues $(i, -i)$, center
- (2,0), $\mathbf{A}$ has eigenvalues (-1,1), saddle node

```
In[33]:= Eigenvalues[{{0,1},{-1,0}}]
```

$\{i, -i\}$

```
In[34]:= Eigenvalues[{{0,1},{1,0}}]
```

$\{-1, 1\}$

```
In[35]:=
```

Exercise. PS4.5.6

Solution. System $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$ has critical points at solutions of $\boldsymbol{f}(\boldsymbol{y}) = 0$,

$$\begin{cases} y_2 = 0 \\ y_1(1 + y_1) = 0 \end{cases}, (y_1, y_2) \in \{(0,0), (-1,0)\}.$$

The Jacobian obtained by linearization is

$$\mathbf{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} 0 & 1 \\ -1 - y_1 & 0 \end{pmatrix}.$$

At critical point:

- $(0,0)$, $\mathbf{A}$ has eigenvalues $(i, -i)$, center
- (-1,0), $\mathbf{A}$ has eigenvalues $(i\sqrt{2}, -i\sqrt{2})$, center

```
In[33]:= Eigenvalues[{{0,1},{-1,0}}]
```

$\{i, -i\}$

```
In[35]:= Eigenvalues[{{0,-2},{1,0}}]
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$\{i\sqrt{2}, -i\sqrt{2}\}$

```
In[36]:=
```

Exercise. PS4.5.7

SOLUTION. System $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$ has critical points at solutions of $\boldsymbol{f}(\boldsymbol{y}) = 0$,

$$\begin{cases} -y_1 + y_2 - y_2^2 = y_2(2 - y_2) = 0 \\ -y_1 - y_2 = 0 \end{cases}, (y_1, y_2) \in \{(0,0), (-2,2)\}.$$

The Jacobian obtained by linearization is

$$\mathbf{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} -1 & 1 - 2y_2 \\ -1 & -1 \end{pmatrix}.$$

At critical point:

- $(0,0)$, $\mathbf{A}$ has eigenvalues $(-1+i, -1-i)$, decaying spiral
- (-2,2), $\mathbf{A}$ has eigenvalues $(-1-\sqrt{3}, \sqrt{3}-1)$, sadle point

```
In[36]:= Eigenvalues[{{-1,1},{-1,-1}}]
```

$\{-1+i, -1-i\}$

```
In[37]:= Eigenvalues[{{-1,-3},{-1,-1}}]
```

$\{-1-\sqrt{3}, \sqrt{3}-1\}$

```
In[38]:=
```

Exercise. PS4.6.2

SOLUTION. System is $\boldsymbol{y}' = \mathbf{A}\boldsymbol{y} + \boldsymbol{r}(t)$,

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + 10 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

Solve eigenproblem $\mathbf{A}\mathbf{X} = \mathbf{X}\boldsymbol{\Lambda}$ to find eigendecomposition

$$\mathbf{X} = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}, \boldsymbol{\Lambda} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Solution of homogeneous system

$$\boldsymbol{y}_h = c_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Seek particular solution by undetermined coefficients

$$\boldsymbol{y}_p = \begin{pmatrix} b_{11}\cos t + b_{12}\sin t \\ b_{21}\cos t + b_{22}\sin t \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \mathbf{B} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Compute

$$\boldsymbol{y}_p' = \mathbf{A}\boldsymbol{y}_p + \boldsymbol{r}(t) \Rightarrow \mathbf{B} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + 10 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \Rightarrow$$

$$\begin{array}{l} -b_{11}\sin t + b_{12}\cos t = -2\cos t + 10\cos t \\ -b_{21}\sin t + b_{22}\cos t = 2\sin t - 10\sin t \end{array} \Rightarrow \mathbf{B} = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}.$$

General solution is

$$\boldsymbol{y} = c_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + 8 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

```
In[44]:= A={{1,1},{3,-1}}; L=DiagonalMatrix[Eigenvalues[A]]
```

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

```
In[42]:= X=Transpose[Eigenvectors[A]]
```

$$\begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}$$

```
In[43]:= Inverse[X]
```

$$\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

```
In[45]:= y={ -c1 Exp[-2t] + c2 Exp[2t] + 8 Sin[t], 3c1 Exp[-2t] + c2 Exp[2t] + 8 Cos[t]}
```

$$\{\mathrm{c1}(-e^{-2t}) + \mathrm{c2}e^{2t} + 8\sin(t), 3\,\mathrm{c1}e^{-2t} + \mathrm{c2}e^{2t} + 8\cos(t)\}$$

```
In[47]:= Simplify[D[y,t] == A.y + 10 {Cos[t],-Sin[t]}]
```

$$\{-8\sin(t) - 10\cos(t), 8\cos(t) - 22\sin(t)\} = \{0, 0\}$$

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In[48]:=
```

Exercise. PS4.6.3

SOLUTION.

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In[1]:=
```

Exercise. PS4.6.4

SOLUTION.

```
In[1]:=
```

Exercise. PS4.6.5

SOLUTION.

```
In[1]:=
```

2 Problems

Problem. PS1.1.16

SOLUTION.

```
In[1]:=
```

Problem. PS1.1.17

SOLUTION.

```
In[1]:=
```

Problem. PS1.1.18

SOLUTION.

```
In[1]:=
```

Problem. PS1.3.22

SOLUTION.

```
In[1]:=
```

3 Projects

3.1 PS2.1.16