Homework 12

1 Exercises

Exercise. PS11.4.11

Solution. The Fourier series of the periodic rectangular wave function $f(x)=2k(u(\pi x)-.5)$, for $x\in[-\pi,\pi)$, $f(x)=f(x+2\pi)$, with $u(x)$ the unit step function is (Example 11.1.1, p. 478)

$$f(x)=\frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 3x+\cdots\right)=\sum_{m=1}^{\infty} b_{2m-1}\sin[(2m-1)x], b_{2m-1}=\frac{4k}{\pi(2m-1)}$$

The Parseval equality states

$$\sum_{m=1}^{\infty} b_{2m-1}^2=\left(\frac{4k}{\pi}\right)^2\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]=\|f\|^2=\int_{-\pi}^{\pi} f(x)^2\,\mathrm{d}x=2\pi k^2\Rightarrow$$

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$$

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In[3]:= S[n_]:=Sum[1/(2m-1)^2,{m,1,n}]; Sinf=N[Pi^2/8.,16]
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1.2337

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In[11]:= Table[N[S[n]],{n,10,100,10}]
```

{1.20872, 1.2212, 1.22537, 1.22745, 1.2287, 1.22953, 1.23013, 1.23058, 1.23092, 1.2312}

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In[13]:= Table[Log10[Abs[N[(S[n]-Sinf)/Sinf]]],{n,10,100,10}]
```

{−1.69363, −1.99439, −2.17043, −2.29535, −2.39225, −2.47143, −2.53838, −2.59637, −2.64752, −2.69327}

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In[14]:=
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Convergence is not rapid due to discontinuity in $f(x)$ at $x=\pm k\pi$

Exercise. PS11.5.7

Solution. The problem $y''+\lambda y=0$, $y(0)=0$, $y(10)=0$ is of Sturm-Liouville type, with the scalar product

$$(f,g)=\int_0^{10} f(x)g(x)\,\mathrm{d}x.$$

For $\lambda=0$, applying boundary conditions to the general solution $y(x)=ax+b$, $y(0)=b=10$, $y(10)=10a=0\Rightarrow a=0$, hence $y(x)=0$ which is not an eigenfunction. For $\lambda>0$, applying boundary conditions to the general solution $y(x)=a\cos(\sqrt{\lambda}\,x)+b\sin(\sqrt{\lambda}\,x)$, $y(0)=a=0$, $y(10)=b\sin(\sqrt{\lambda}\,10)=0\Rightarrow\sqrt{\lambda}10=k\pi\Rightarrow\lambda_k=(k\pi/10)^2$ are eigenvalues, with associated eigenfunctions $y_k(x)=\sin(k\pi x/\sqrt{10})$. For $\lambda<0$, applying boundary conditions to the general solution $y(x)=ae^{\sqrt{\lambda}x}+be^{-\sqrt{\lambda}x}$, $y(0)=a+b=0$, $y(10)=aC+b/C=0$, leads to a homogeneous linear system with principal determinant

$$\Delta=\begin{vmatrix}1 & 1\\ C & 1/C\end{vmatrix}=\frac{1}{C}-C$$

that would have to be null in order to obtain a non-trivial solution. This occurs for $C=1=e^{\sqrt{\lambda}10}\Rightarrow\lambda=0$, a contradiction. Hence the only eigenvalue, eigenfunction pairs are

$$\lambda_k=(k\pi/10)^2, y_k(x)=\sin(k\pi x/\sqrt{10}).$$

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In[1]:=
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Exercise. PS1.1.7

SOLUTION.

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In[1]:=
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Exercise. PS1.1.8

SOLUTION.

```
In[1]:=
```

Exercise. PS1.2.2

SOLUTION.

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In[1]:=
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Exercise. PS1.2.3

SOLUTION.

```
In[1]:=
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Exercise. PS1.2.4

SOLUTION.

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In[1]:=
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Exercise. PS1.3.5

SOLUTION.

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In[1]:=
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2 Problems

Problem. PS11.6.1

SOLUTION. The function $f(x) = 63x^5 - 90x^3 + 35x$, has the Fourier-Legendre series expansion

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x),$$

with

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x)\, P_m(x) \mathrm{d}x,$$

and $P_m(x)$ the Legendre polynomials

```
In[16]:= p=Table[LegendreP[m,x],{m,0,5}]
```

$$\left\{1, x, \frac{1}{2}(3\,x^2-1), \frac{1}{2}(5\,x^3-3\,x), \frac{1}{8}(35\,x^4-30\,x^2+3), \frac{1}{8}(63\,x^5-70\,x^3+15\,x)\right\}$$

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In[17]:=
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Since $f(x)$ has only odd powers less than $m=5$, the expansion is

$$f(x) = a_1 P_1(x) + a_3 P_3(x) + a_5 P_5(x).$$

One can compute a_m directly, but simple observations lead to a quicker result

$$a_5 = 8,\ f(x) - a_5 P_5(x) =$$

```
In[17]:= f[x_]=63 x^5 - 90 x^3 + 35 x;
         Table[ (2m+1)/2 Integrate[f[x] LegendreP[m,x],{x,-1,1}],{m,1,5}]
```

$\{8, 0, -8, 0, 8\}$

```
In[18]:= f[x]-8 LegendreP[5,x]
```

$20\,x - 20\,x^3$

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In[20]:= Expand[f[x]- 8 LegendreP[5,x] + 8 LegendreP[3,x]]
```

$8\,x$

In[21]:=

Problem. PS1.1.17
Solution.

In[1]:=

Problem. PS1.1.18
Solution.

In[1]:=

Problem. PS1.3.22
Solution.

In[1]:=

3 Projects

3.1 PS2.1.16