MATH528 Oct 17, 2018

Midterm examination

Instructions: Exam is closed book. Present your answers legibly, show and briefly comment steps in your solution.

1. Find the general solution of the following ODEs:

   a) $y' + 2.5\,y = 1.6x$

   b) $y' - 0.4y = 29\sin x$

   c) $25\,yy' - 4x = 0$

   Classify each equation. (3×2 points = 6 points)

   Solution.

   a) $y' + 2.5\,y = 1.6x$ is a linear, first-order, explicit, inhomogeneous ODE. Solution of homogeneous equation $y' + 2.5y = 0$ is $y(x) = ce^{-2.5x}$. Apply variation of parameters, $y(x) = c(x)e^{-2.5x}$ to find general solution of inhomogeneous equation

   $$y' + 2.5y = c'e^{-2.5x} = 1.6x \Rightarrow c(x) = 1.6x\,e^{2.5x} \Rightarrow c(x) = 1.6\int x\,e^{2.5x}\,\mathrm{d}x + a,$$

   and integration by parts $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$, $u = x$, $\mathrm{d}v = \mathrm{d}\left(\frac{1}{2.5}e^{2.5x}\right)$ gives

   $$c(x) = \frac{1.6}{2.5}\left(x\,e^{2.5x} - \int e^{2.5x}\,\mathrm{d}x\right) + a = \frac{1.6}{2.5}\left(x - \frac{1}{2.5}\right)e^{2.5x} + a = 1.6(0.4x - 0.16)e^{2.5x} + a.$$

   General solution is

   $$y(x) = 1.6(0.4x - 0.16) + a\,e^{-2.5x}.$$

   b) $y' - 0.4y = 29\sin x$ is a linear, first-order, explicit, inhomogeneous ODE. Solution of homogeneous equation $y' - 0.4y = 0$ is $y = ce^{0.4x}$. Use method of undetermined coefficients to seek general solution as

   $$y(x) = ce^{0.4x} + a\sin x + b\cos x.$$

   Replacing,

   $$y' - 0.4y = (a - 0.4b)\cos x - (b + 0.4a)\sin x = 29\sin x \Rightarrow a - 0.4b = 0, b + 0.4a = -29$$

   $$a = 0.4b, 1.16b = 29 \Rightarrow b = \frac{-29}{1.16} = -25, a = \frac{-29}{2.9} = -10$$

   and find the general solution

   $$y(x) = ce^{0.4x} - 10\sin x - 25\cos x.$$

   c) $25\,yy' - 4x = 0$, is a non-linear, first-order, implicit, separable, inhomogeneous ODE. Direct integration gives

   $$\frac{25}{2}\,y^2 - 2x^2 = c \Rightarrow y = \pm\frac{1}{5}\sqrt{c + 4x^2}\,.$$

2. Solve the IVP $y''' - y'' - y' + y = 0$, $y(0) = 0, y'(0) = 1, y''(0) = 0$. (3 points)

   Solution. Third-order, constant-coefficient, linear ODE. Trying solution of form $y(x) = e^{rx}$, the characteristic equation

   $$r^3 - r^2 - r + 1 = (r - 1)(r^2 - 1) = 0$$

   results with roots $r_{1,2} = 1$ (double root), $r_3 = -1$. The general solution is

   $$y(x) = c_1e^x + c_2x\,e^x + c_3e^{-x}.$$

Applying initial conditions gives

$$c_1+c_3=0, c_1+c_2-c_3=1, c_1+2c_2+c_3=0 \Rightarrow c_1=\frac{1}{2}, c_2=0, c_3=-\frac{1}{2}$$

so the solution is

$$y(x)=\frac{1}{2}(e^x-e^{-x})=\sinh x$$

3. Find the general solution and determine the type of critical points of the system

$$\begin{array}{l} y_1'=3y_1+4y_2 \\ y_2'=3y_1+2y_2 \end{array}.$$

(3 points)

Solution. Solve the eigenproblem, $AX=X\Lambda$

$$A=\left(\begin{array}{cc} 3 & 4 \\ 3 & 2 \end{array}\right), \det(A-\lambda I)=(3-\lambda)(2-\lambda)-12=\lambda^2-5\lambda-6=(\lambda-6)(\lambda+1)=0 \Rightarrow \lambda_1=-1, \lambda_2=6.$$

$$A-\lambda_1 I=\left(\begin{array}{cc} 4 & 4 \\ 3 & 3 \end{array}\right) \Rightarrow x_1=\left(\begin{array}{c} 1 \\ -1 \end{array}\right), A-\lambda_2 I=\left(\begin{array}{cc} -3 & 4 \\ 3 & -4 \end{array}\right) \Rightarrow x_2=\left(\begin{array}{c} 4 \\ 3 \end{array}\right),$$

$$\Lambda=\left(\begin{array}{cc} -1 & 0 \\ 0 & 6 \end{array}\right), X=\left(\begin{array}{cc} x_1 & x_2 \end{array}\right)=\left(\begin{array}{cc} 1 & 4 \\ -1 & 3 \end{array}\right), X^{-1}=\frac{1}{7}\left(\begin{array}{cc} 3 & -4 \\ 1 & 1 \end{array}\right).$$

The system $y'=Ay=X\Lambda X^{-1}y$, is equivalent to $z'=\Lambda z$, with $z=X^{-1}y$ and solution

$$z(x)=\left(\begin{array}{c} c_1 e^{-x} \\ c_2 e^{6x} \end{array}\right), y=Xz=\left(\begin{array}{cc} 1 & 4 \\ -1 & 3 \end{array}\right)\left(\begin{array}{c} c_1 e^{-x} \\ c_2 e^{6x} \end{array}\right)=\left(\begin{array}{c} c_1 e^{-x}+4c_2 e^{6x} \\ -c_1 e^{-x}+3c_2 e^{6x} \end{array}\right)=\left(\begin{array}{cc} e^{-x} & 4e^{6x} \\ -e^{-x} & 3e^{6x} \end{array}\right)\left(\begin{array}{c} c_1 \\ c_2 \end{array}\right).$$

Since $\det(A)\neq 0$, the only solution of $y'=0$ is $y=0$, and the origin is a saddle point (one positive, one negative eigenvalue of A).