MATH528 mLab03: Investigating ODE models

First-order linear ODE models

Heating and cooling of a building (PS1.5.32)

Building ambient temperature

In[1]:=
```
Ta[t_] = A - c Cos[2 Pi t / 24]
```

Out[1]= $A - c\,\text{Cos}\left[\frac{\pi t}{12}\right]$

Rate of change of temperature in building

In[2]:=
```
ODE = T'[t] == k1 (T[t] - Ta[t]) + k2 (T[t] - Tw) + P
```

Out[2]= $T'[t] == P + k_1 \left(-A + c\,\text{Cos}\left[\frac{\pi t}{12}\right] + T[t]\right) + k_2 \left(-T_w + T[t]\right)$

In[3]:=
```
h[t_] = -(k1 + k2) t
```

Out[3]= $t\left(-k_1 - k_2\right)$

In[4]:=
```
r[t_] = k1 Ta[t] - k2 Tw + P
```

Out[4]= $P + \left(A - c\,\text{Cos}\left[\frac{\pi t}{12}\right]\right) k_1 - k_2 T_w$

In[5]:=
```
TGenSol[t_] = FullSimplify[Exp[-h[t]] (Integrate[Exp[h[t]] r[t], t] + B)]
```

Out[5]= $B\, e^{t\,(k_1+k_2)} - \frac{12\, c\, \pi\, \text{Sin}\left[\frac{\pi t}{12}\right] k_1}{\pi^2 + 144\left(k_1 + k_2\right)^2} + \frac{144\, c\, \text{Cos}\left[\frac{\pi t}{12}\right] k_1 \left(k_1 + k_2\right)}{\pi^2 + 144\left(k_1 + k_2\right)^2} - \frac{P + A k_1 - k_2 T_w}{k_1 + k_2}$

In[6]:=
```
TGenSol[0]
```

Out[6]= $B + \frac{144\, c\, k_1 \left(k_1 + k_2\right)}{\pi^2 + 144\left(k_1 + k_2\right)^2} - \frac{P + A k_1 - k_2 T_w}{k_1 + k_2}$

In[7]:=
```
Bsol = Solve[TGenSol[0] == T0, B][[1, 1]]
```

Out[7]= $B \to -\frac{144\, c\, k_1 \left(k_1 + k_2\right)}{\pi^2 + 144\left(k_1 + k_2\right)^2} + T_0 + \frac{P + A k_1 - k_2 T_w}{k_1 + k_2}$

In[8]:=
```
T[t_] = FullSimplify[TGenSol[t] /. Bsol]
```

Out[8]= $-\frac{12\, c\, \pi\, \text{Sin}\left[\frac{\pi t}{12}\right] k_1}{\pi^2 + 144\left(k_1 + k_2\right)^2} + \frac{144\, c\, \text{Cos}\left[\frac{\pi t}{12}\right] k_1 \left(k_1 + k_2\right)}{\pi^2 + 144\left(k_1 + k_2\right)^2} - \frac{P + A k_1 - k_2 T_w}{k_1 + k_2} + e^{t\,(k_1+k_2)} \left(-\frac{144\, c\, k_1 \left(k_1 + k_2\right)}{\pi^2 + 144\left(k_1 + k_2\right)^2} + T_0 + \frac{P + A k_1 - k_2 T_w}{k_1 + k_2}\right)$

In[9]:=
```
T[0]
```

Out[9]= T_0

In[10]:=
```
Manipulate[T[t] /. {T₀ → T0}, {T0, 65, 75}]
```

T0

Out[10]=

$$-\left(\left(12\, c\, \pi\, \mathrm{Sin}\left[\frac{1}{12}\pi\, \text{Charting`Private`pvar\$11021}\right] k_1\right) \Big/ \left(\pi^2 + 144\left(k_1+k_2\right)^2\right)\right) + \left(144\, c\, \mathrm{Cos}\left[\frac{1}{12}\pi\, \text{Charting`Private`pvar\$11021}\right] k_1 \left(k_1+k_2\right)\right) \Big/ \left(\pi^2 + 144\left(k_1+k_2\right)^2\right) - \frac{P + A\, k_1 - k_2\, T_w}{k_1+k_2} + e^{\text{Charting`Private`pvar\$11021}\,(k_1+k_2)} \left(65 - \frac{144\, c\, k_1 \left(k_1+k_2\right)}{\pi^2 + 144\left(k_1+k_2\right)^2} + \frac{P + A\, k_1 - k_2\, T_w}{k_1+k_2}\right)$$

In[11]:=
```
Manipulate[T[t] /. {T₀ → T0, Tw → Tw}, {T0, 45, 95}, {Tw, 65, 72}]
```

T0

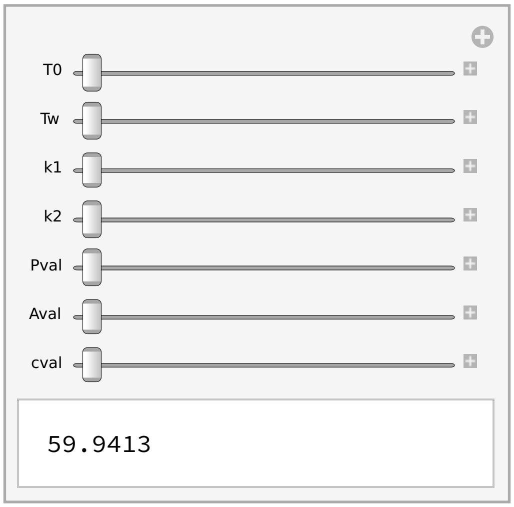
Tw

Out[11]=

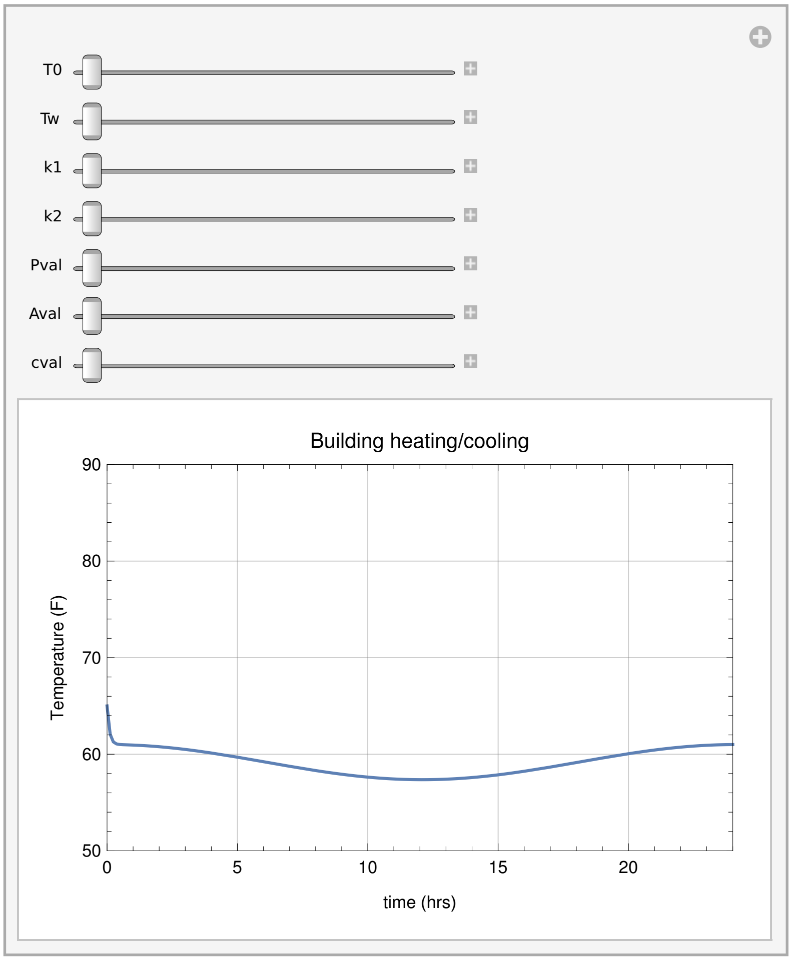
$$-\frac{P + A\, k_1 - 65\, k_2}{k_1+k_2} - \frac{2.68589\, c\, k_1}{\pi^2 + 144\left(k_1+k_2\right)^2} - \frac{143.634\, c\, k_1 \left(k_1+k_2\right)}{\pi^2 + 144\left(k_1+k_2\right)^2} + e^{11.7276\,(k_1+k_2)} \left(45 + \frac{P + A\, k_1 - 65\, k_2}{k_1+k_2} - \frac{144\, c\, k_1 \left(k_1+k_2\right)}{\pi^2 + 144\left(k_1+k_2\right)^2}\right)$$

In[12]:=
```
Manipulate[T[t] /. {T_0 → T0, T_w → Tw, k_1 → k1, k_2 → k2, P → Pval, A → Aval, c → cval},
  {T0, 65, 75}, {Tw, 65, 72}, {k1, -1, -0.1}, {k2, -10, -1},
  {Pval, 1, 10}, {Aval, 0, 100}, {cval, 20, 40}]
```

Out[12]=



In[13]:=
```
Manipulate[
 Plot[
  T[t] /. {T_0 → T0, T_w → Tw, k_1 → k1, k_2 → k2, P → Pval, A → Aval, c → cval}, {t, 0, 24},
  Axes → False, Frame → True, FrameLabel → {"time (hrs)", "Temperature (F)"},
  GridLines → Automatic, PlotLabel → "Building heating/cooling",
  PlotRange → {{0, 24}, {50, 90}}],
 {T0, 65, 75}, {Tw, 65, 72}, {k1, -1, -0.1}, {k2, -10, -1},
 {Pval, 1, 10}, {Aval, 0, 100}, {cval, 20, 40}]
```

Out[13]=


Continual harvesting of renewable resources (PS1.5.36)

Logistic equation model

In[14]:=
```
ODE = y'[t] == (A - H) y[t] - B y[t]^2
```

Out[14]= $y'[t] == (A - H)\, y[t] - B\, y[t]^2$

In[15]:=
```
iCond = y[0] == y0
```

Out[15]= y[0] == y0

In[16]:=
```
sol[t_, A_, B_, H_, y0_] = FullSimplify[y[t] /. DSolve[{ODE, iCond}, y[t], t][[1, 1]]]
```

... Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

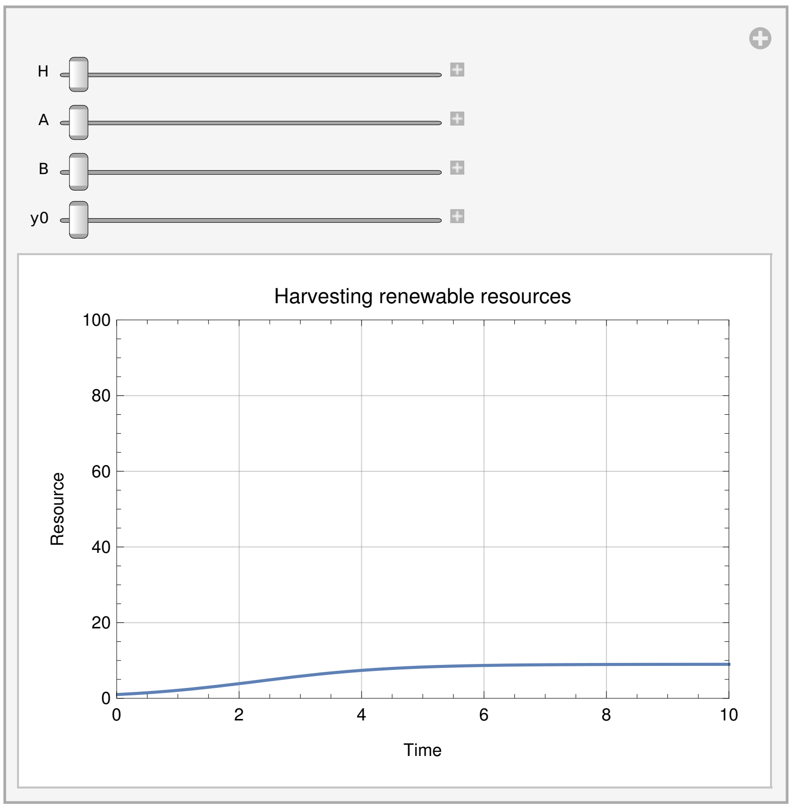
Out[16]=
$$\frac{(A - H)\, y0}{B\, y0 + e^{(-A+H)\, t}\, (A - H - B\, y0)}$$

In[17]:=
```
sol[0]
```

Out[17]=
```
sol[0]
```

In[18]:=
```
Manipulate[Plot[sol[t, A, B, H, y0], {t, 0, 10}, PlotRange → {{0, 10}, {0, 100}},
  Axes → False, Frame → True, FrameLabel → {"Time", "Resource"},
  GridLines → Automatic, PlotLabel → "Harvesting renewable resources"] ,
 {H, 0.1, A}, {A, 1, 10}, {B, 0.1, 2}, {y0, 1, 100}]
```

Out[18]=

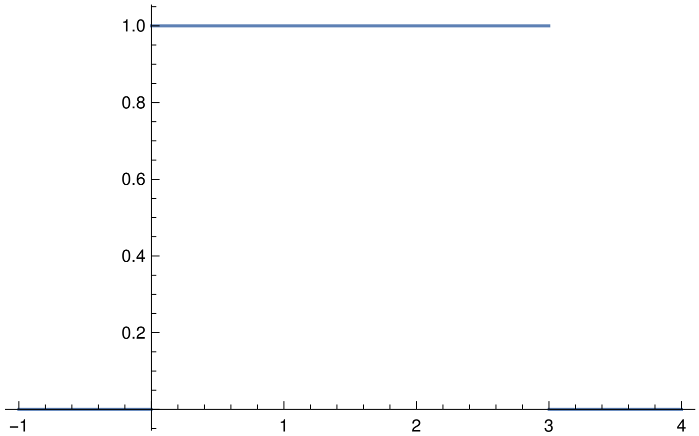


Intermittent harvesting renewable resources (PS1.5.36-38)

The step function HeavisidePi is useful for this problem. Here’s a plot. Notice the scaling of the argument to obtain the desired 3-year harvesting period.

In[19]:=
```
Plot[HeavisidePi[t/3 - .5], {t, -1, 4}]
```

Out[19]=



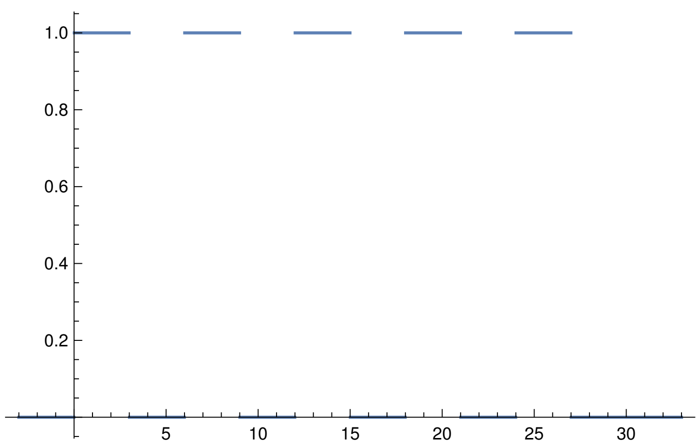
Now, construct a sum of HeavisidePi step functions to cover a time span of 30 years

In[20]:=
```
Hi[t_] = Sum[HeavisidePi[t/3 - .5 - k], {k, 0, 9, 2}]
```

Out[20]= $\text{HeavisidePi}\left[0.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[2.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[4.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[6.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[8.5-\frac{t}{3}\right]$

In[21]:=
```
Plot[Hi[t], {t, -3, 33}]
```

Out[21]=



Redefine the ODE to model intermittent harvesting

In[22]:=
```
ODE = y'[t] == (A - H Hi[t]) y[t] - B y[t]^2
```

Out[22]= $y'[t] == \left(A-H\left(\text{HeavisidePi}\left[0.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[2.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[4.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[6.5-\frac{t}{3}\right]+\text{HeavisidePi}\left[8.5-\frac{t}{3}\right]\right)\right)y[t]-B\,y[t]^2$

In[23]:=
```
iCond = y[0] == y0
```

Out[23]= y[0] == y0

Analytical solution, though possible, is exceedingly complicated. A numerical solution is readily obtainable

In[24]:=
```
soli[t_, Av_, Bv_, Hv_, y0v_] := y[t] /.
  NDSolve[{ODE, iCond} /. {A → Av, B → Bv, H → Hv, y0 → y0v} , y[t], {t, 0, 30}][[1, 1]]
```

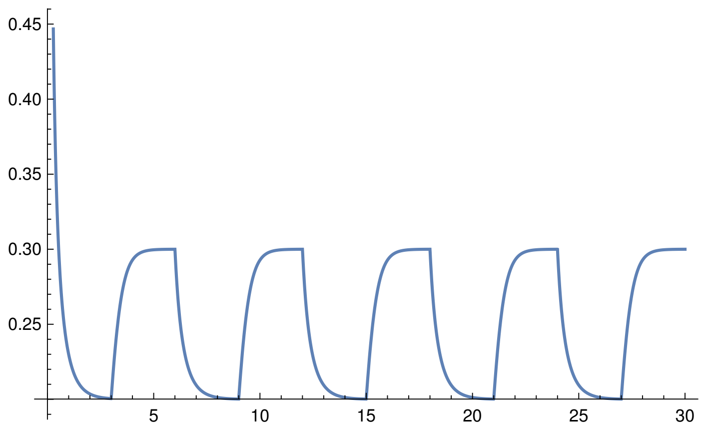
The numerical solution is represented by an interpolation

In[25]:=
```
soli[t, 3., 10., 1., 5.]
```

Out[25]=
```
InterpolatingFunction[ Domain: {{0., 30.}} Output: scalar ][t]
```

In[26]:=
```
Plot[Evaluate[soli[t, 3., 10., 1., 5.]], {t, 0, 30}]
```

Out[26]=



In[27]:=
```
Manipulate[Plot[{Evaluate[soli[t, A, B, H, y0]], sol[t, A, B, H, y0]},
  {t, 0, 30}, PlotRange → {{0, 30}, {0, 25}}, Axes → False, Frame → True,
  FrameLabel → {"Time", "Resource"}, GridLines → Automatic,
  PlotStyle → {Black, Red}, PlotLabel → "Harvesting renewable resources"] ,
 {H, 0.1, A}, {A, 1, 10}, {B, 0.05, 1}, {{y0, 10}, 0, 25}]
```

Out[27]=