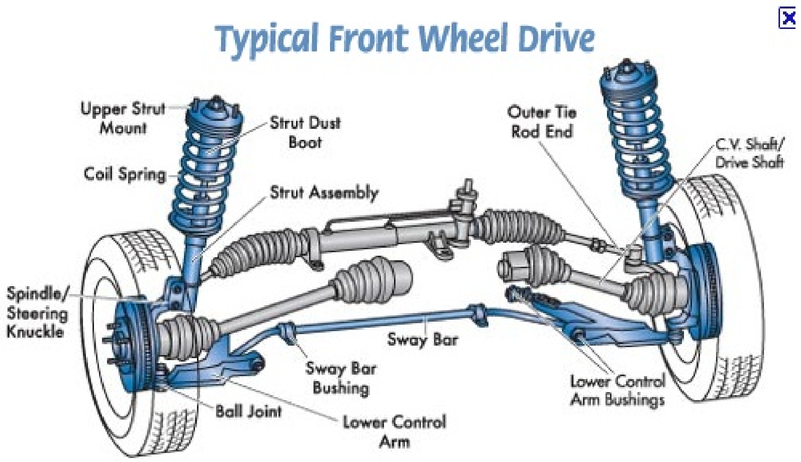
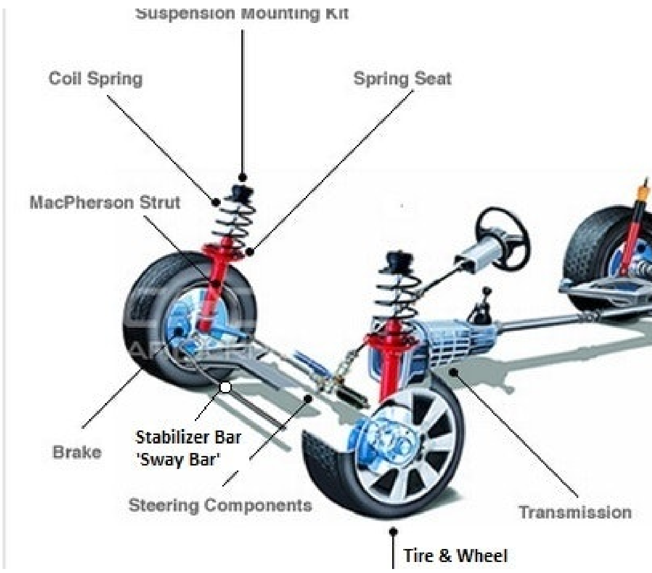
Mini-Lab 06

In a two-mini-lab sequence, we study a realistic model: the dynamics of a car. In this first mini-lab we solve some simpler lead-up problems involving point masses and springs





1 Warm-up problems: lossless oscillators in series

1.1 Single spring

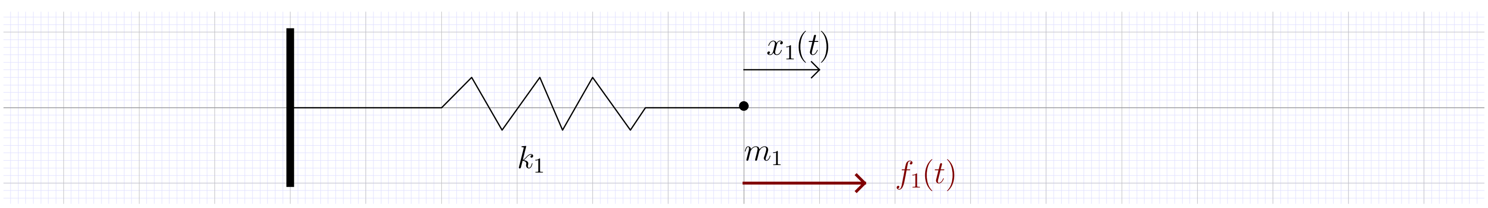


Figure 1. Two springs connected in series

Consider the model resulting from conservation of momentum

$$\begin{aligned} \dot{u}_1 &= -\frac{k_1}{m_1} x_1 \\ \dot{x}_1 &= u_1 \end{aligned}$$

that can expressed in vector form as

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}, \boldsymbol{y} = \begin{pmatrix} u_1 \\ x_1 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} 0 & -\frac{k_1}{m_1} \\ 1 & 0 \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

Define the system matrix in Mathematica

```
In[1]:= A={{0,-k1/m1},{1,0}};
        MatrixForm[A]
```

$$\begin{pmatrix} 0 & -\frac{\text{k1}}{\text{m1}} \\ 1 & 0 \end{pmatrix}$$

```
In[2]:=
```

Define the number of components in the dependent variable $\boldsymbol{y}(t)$ and the forcing term f

```
In[2]:= y[t_]={y1[t],y2[t]};
        f[t_]={f1[t],0};
        SYS = y'[t] == A . y[t] + f[t];
        MatrixForm[A . y[t] + f[t]]
```

$$\begin{pmatrix} \mathrm{f1}(t) - \frac{\mathrm{k1y2}(t)}{\mathrm{m1}} \\ \mathrm{y1}(t) \end{pmatrix}$$

```
In[3]:=
```

Now, look at the solutions for some typical initial conditions and forcing terms:

Case 1. No initial velocity, point masses out of equilibrium position, no forcing. Define a set of replacement rules for the system parameters defining this case

```
In[3]:= F1[t_]=0;
        case1 = {f1->F1,m1->1,k1->1}
```

$\{\mathrm{f1}\to\mathrm{F1}, \mathrm{m1}\to 1, \mathrm{k1}\to 1\}$

```
In[4]:=
```

The system matrix for this case:

```
In[4]:= MatrixForm[A /. case1]
```

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

```
In[5]:=
```

Find the eigenvalues

```
In[5]:= MatrixForm[DiagonalMatrix[Eigenvalues[A /. case1]]]
```

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

```
In[6]:= MatrixForm[Eigenvectors[A /. case1]]
```

$$\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

```
In[7]:=
```

As expected, eigenvalues are purely imaginary indicating oscillations with no dampening. Solve the system for some chosen set of initial conditions

```
In[7]:= iCond = {0,1};
        F1[t_]=0;
        MatrixForm[SYS /. case1]
```

$\{\mathrm{y1}'(t), \mathrm{y2}'(t)\} = \{-\mathrm{y2}(t), \mathrm{y1}(t)\}$

```
In[8]:= GenSol[t_] = DSolveValue[ SYS /. case1, y[t],t]
```

$\{c_1\cos(t) - c_2\sin(t), c_1\sin(t) + c_2\cos(t)\}$

```
In[9]:= c={C[1],C[2]};
        sol[t_] = GenSol[t] /. Solve[GenSol[0] == iCond,c]
```

$\begin{pmatrix} -\sin(t) & \cos(t) \end{pmatrix}$

```
In[11]:= SingleOscPlt1 = ParametricPlot[sol[t],{t,0,2Pi},Axes->True,Frame->True,
         FrameLabel->{"y1","y2"}];
         Export["/home/student/courses/MATH528/L05Fig01.pdf",SingleOscPlt1]
```

/home/student/courses/MATH528/L05Fig01.pdf

In[12]:=

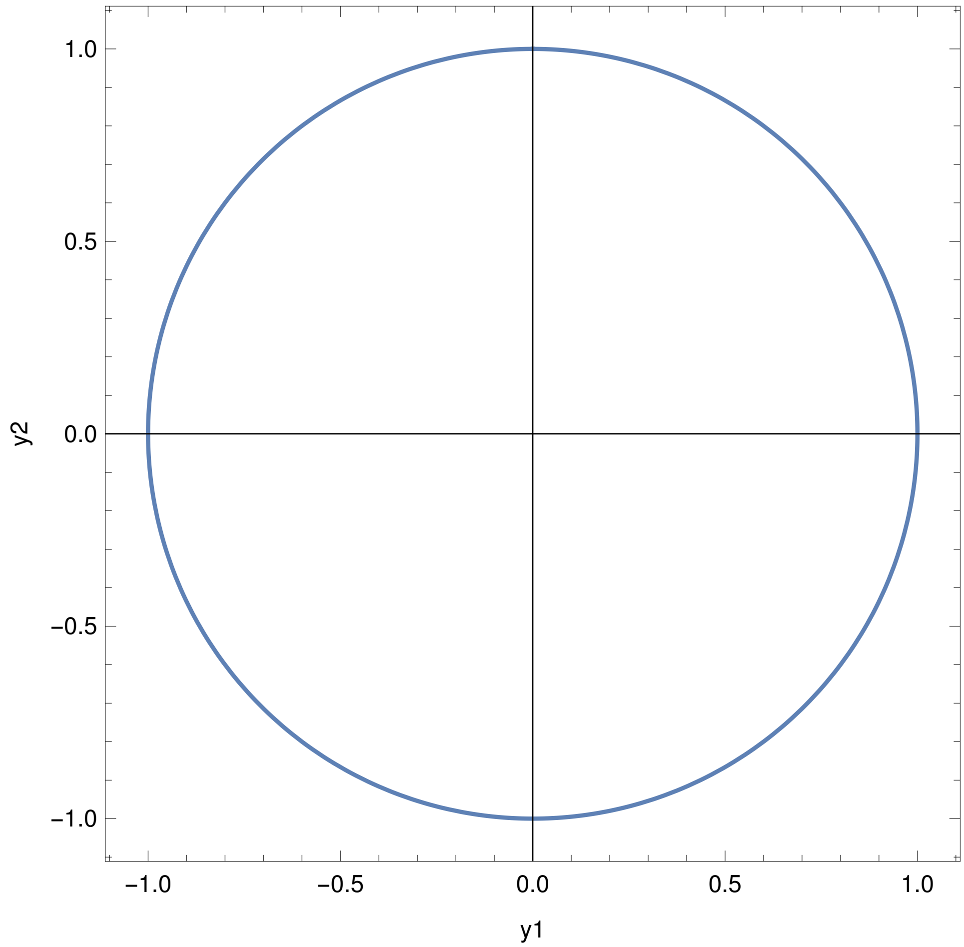


Figure 2. Phase plane trajectory for unforced harmonic oscillator

Case 2. No initial velocity, point masses at equilibrium, with forcing

```
In[12]:= F1[t]=Sin[2t];
         case2 = {f1->F1,m1->1,k1->1};
         GenSol[t_] = DSolveValue[ SYS /. case1, y[t],t]
```

$$\left\{-c_2 \sin(t) + c_1 \cos(t) + \frac{2\sin^4(t)}{3} + \cos(t)\left(-\frac{\cos(t)}{2} - \frac{1}{6}\cos(3t)\right), c_1 \sin(t) + c_2 \cos(t) - \frac{2}{3}\sin^3(t)\cos(t) + \sin(t)\left(-\frac{\cos(t)}{2} - \frac{1}{6}\cos(3t)\right)\right\}$$

```
In[17]:= c={C[1],C[2]};
         sol[t_] = TrigReduce[GenSol[t] /. Solve[GenSol[0] == iCond,c]]
```

$$\begin{pmatrix} \frac{1}{3}(2\cos(t) - 2\cos(2t) - 3\sin(t)) & \frac{1}{3}(3\cos(t) + 2\sin(t) - \sin(2t)) \end{pmatrix}$$

```
In[18]:= SingleOscPlt2 = ParametricPlot[sol[t],{t,0,2Pi},Axes->True,Frame->True,
         FrameLabel->{"y1","y2"}];
         Export["/home/student/courses/MATH528/L05Fig02.pdf",SingleOscPlt2]
```

/home/student/courses/MATH528/L05Fig02.pdf

In[19]:=

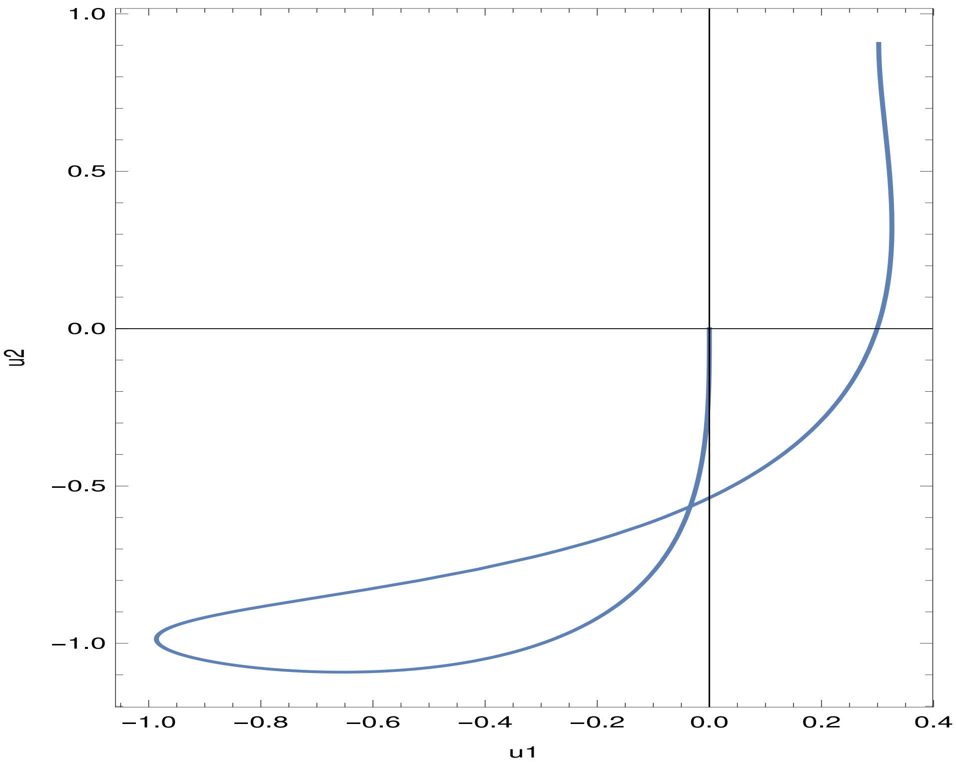


Figure 3. Phase trajectory for forced oscillator

1.2 Two springs in series

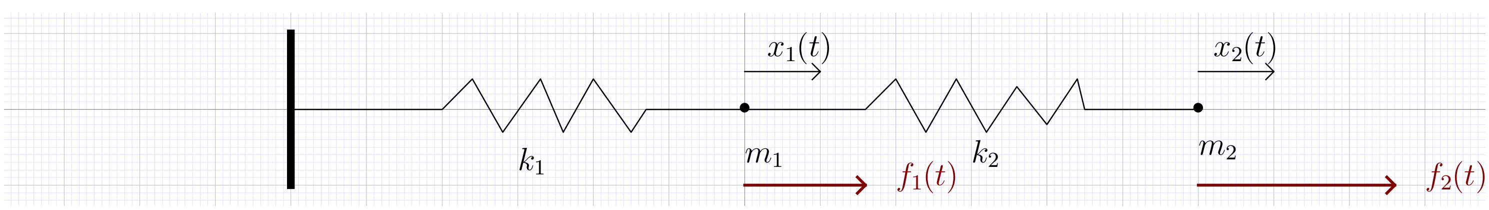


Figure 4. Two springs connected in series

Consider the model resulting from conservation of momentum

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{u}_1 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) + f_1 \\
\dot{u}_2 &= -\frac{k_2}{m_2}(x_2 - x_1) + f_2
\end{aligned}$$

that can expressed in vector form as

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}, \boldsymbol{y} = \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} 0 & 0 & -\frac{k_1 + k_2}{m_1} & \frac{k_2}{m_1} \\ 0 & 0 & -\frac{k_2}{m_2} & -\frac{k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{pmatrix}$$

Define the system matrix in Mathematica

```
In[1]:= A={{0,0,-(k1+k2)/m1,k2/m1},{0,0,k2/m2,-k2/m2},{1,0,0,0},{0,1,0,0}};
        MatrixForm[A]
```

$$\begin{pmatrix} 0 & 0 & -\frac{\text{k1}+\text{k2}}{\text{m1}} & \frac{\text{k2}}{\text{m1}} \\ 0 & 0 & \frac{\text{k2}}{\text{m2}} & -\frac{\text{k2}}{\text{m2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[2]:=

Define the number of components in the dependent variable $\boldsymbol{y}(t)$ and the forcing term f

```
In[2]:= y[t_]={y1[t],y2[t],y3[t],y4[t]};
        f[t_]={f1[t],f2[t],0,0};
        SYS = y'[t] == A . y[t] + f[t];
        MatrixForm[A . y[t] + f[t]]
```

$$\begin{pmatrix} \text{f1}(t) - \frac{(\text{k1}+\text{k2})\text{y3}(t)}{\text{m1}} + \frac{\text{k2y4}(t)}{\text{m1}} \\ \text{f2}(t) + \frac{\text{k2y3}(t)}{\text{m2}} - \frac{\text{k2y4}(t)}{\text{m2}} \\ \text{y1}(t) \\ \text{y2}(t) \end{pmatrix}$$

In[3]:=

Now, look at the solutions for some typical initial conditions and forcing terms:

Case 1. No initial velocity, point masses out of equilibrium position, no forcing. Define a set of replacement rules for the system parameters defining this case

```
In[3]:= case1 = {f1->F1,f2->F2,m1->1,m2->1,k1->1,k2->1}
```

$\{\text{f1}\to\text{F1}, \text{f2}\to\text{F2}, \text{m1}\to 1, \text{m2}\to 1, \text{k1}\to 1, \text{k2}\to 1\}$

In[4]:=

The system matrix for this case:

```
In[4]:= MatrixForm[A /. case1]
```

$$\begin{pmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[5]:=

Find the eigenvalues

```
In[5]:= MatrixForm[DiagonalMatrix[Eigenvalues[A /. case1]]]
```

$$\begin{pmatrix} i\sqrt{\frac{1}{2}(3+\sqrt{5})} & 0 & 0 & 0 \\ 0 & -i\sqrt{\frac{1}{2}(3+\sqrt{5})} & 0 & 0 \\ 0 & 0 & i\sqrt{\frac{1}{2}(3-\sqrt{5})} & 0 \\ 0 & 0 & 0 & -i\sqrt{\frac{1}{2}(3-\sqrt{5})} \end{pmatrix}$$

In[6]:=

As expected, eigenvalues are purely imaginary indicating oscillations with no dampening. Solve the system for some chosen set of initial conditions. As the system matrix size increases analytical expressions for the eigenvalues, eigenvectors become complicated or impossible to find.

```
In[8]:= iCond = {0,0,1,2};
        {F1[t_],F2[t_]}={0,0};
        MatrixForm[A . y[t] + f[t] /. case1]
```

$$\begin{pmatrix} \mathrm{y4}(t) - 2\mathrm{y3}(t) \\ \mathrm{y3}(t) - \mathrm{y4}(t) \\ \mathrm{y1}(t) \\ \mathrm{y2}(t) \end{pmatrix}$$

```
In[9]:=
```

We turn to numerical evaluation of the solution to the system of ODEs

```
In[20]:= sol[t_]=NDSolveValue[ {y'[t]==A . y[t] + f[t] /. case1,y[0]==iCond}, y[t],{t,0,
         32Pi}];
         DoubleOscPlt1 = ParametricPlot[sol[t][[1;;2]],{t,0,32Pi},Axes->True,Frame-
         >True,FrameLabel->{"u1","u2"}];
         Export["/home/student/courses/MATH528/L05Fig03.pdf",DoubleOscPlt1]
```

/home/student/courses/MATH528/L05Fig03.pdf

```
In[21]:=
```

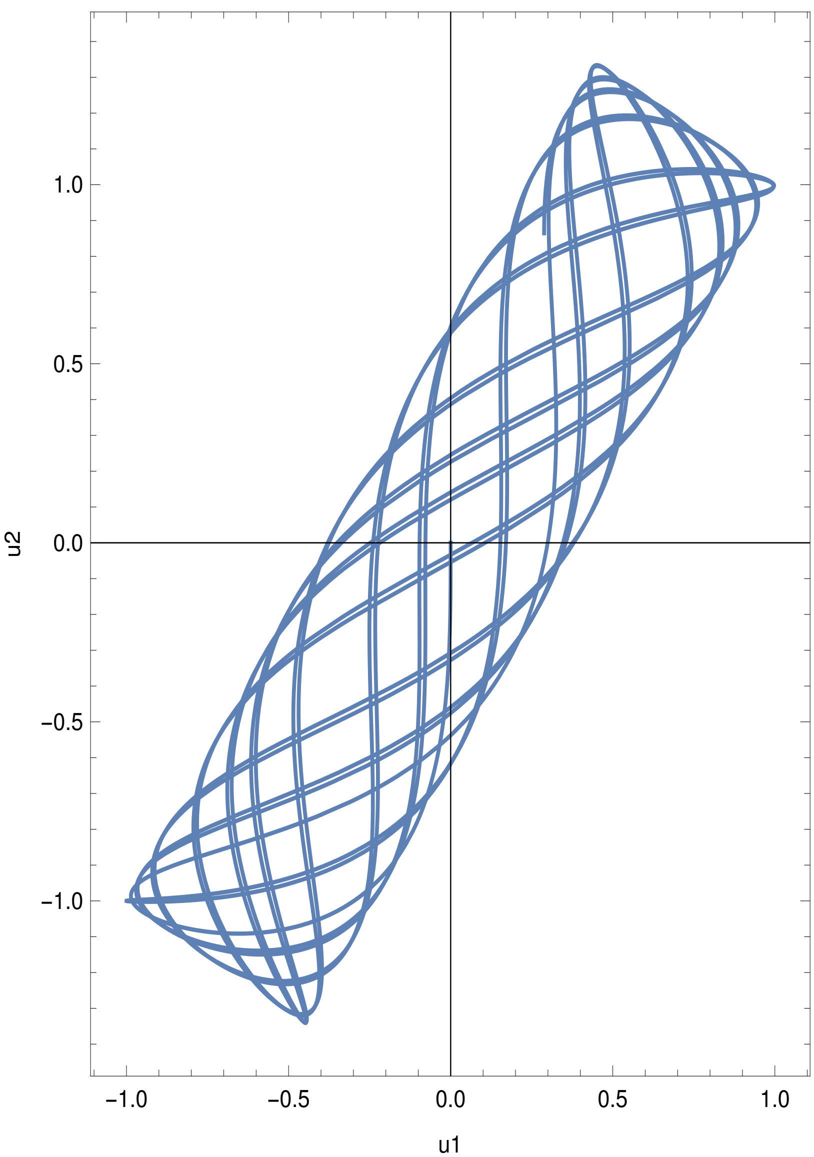


Figure 5. Phase plane of velocities of the double spring

2 Car suspension dynamical system

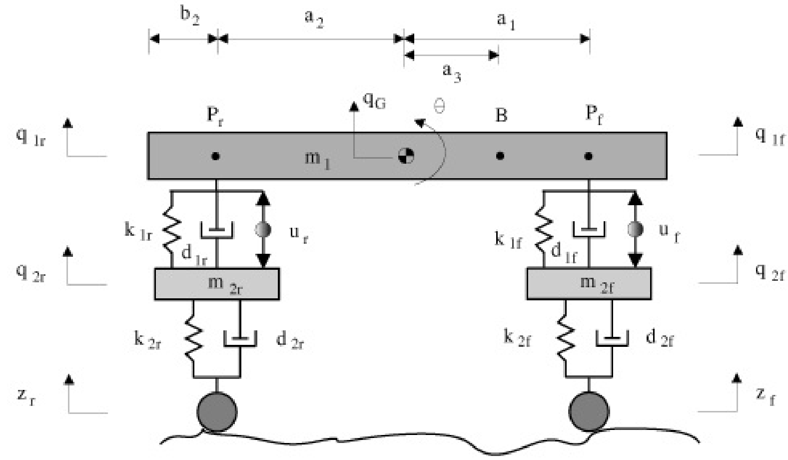


Figure 1. Half-vehicle model with rigid body.

- Terrain modeled by known functions $z_f(t), z_r(t)$
- Car suspension described by: $q_{1f}(t), q_{2f}(t), q_{1r}(t), q_{2r}(t)$
- Dynamics described by Newton's law for linear and angular momentum

$$
\begin{aligned}
m_{2r}\ddot{q}_{2r} &= k_{1r}(q_{1r}-q_{2r}) - k_{2r}(q_{2r}-z_r) + d_{1r}(\dot{q}_{1r}-\dot{q}_{2r}) - d_{2r}(\dot{q}_{2r}-\dot{z}_r) \\
m_{2f}\ddot{q}_{2f} &= k_{1f}(q_{1f}-q_{2f}) - k_{2f}(q_{2f}-z_f) + d_{1f}(\dot{q}_{1f}-\dot{q}_{2f}) - d_{2f}(\dot{q}_{2f}-\dot{z}_f) \\
m_1\frac{\ddot{q}_{1r}+\ddot{q}_{1f}}{2} &= -k_{1r}(q_{1r}-q_{2r}) - k_{1f}(q_{1f}-q_{2f}) - d_{1r}(\dot{q}_{1r}-\dot{q}_{2r}) - d_{1f}(\dot{q}_{1f}-\dot{q}_{2f}) \\
I\frac{\ddot{q}_{1f}-\ddot{q}_{1r}}{2a} &= a[k_{1r}(q_{1r}-q_{2r}) + d_{1r}(\dot{q}_{1r}-\dot{q}_{2r})] - a[k_{1f}(q_{1f}-q_{2f}) + d_{1f}(\dot{q}_{1f}-\dot{q}_{2f})]
\end{aligned}
$$