Definition. *A model is a simplified representation of a phenomenon of interest.*

Examples:

- Physics model. An elastic solid is a simplified representation of the behavior of the atoms in an aluminum bar
- Anatomy model. A mouse lung is a simplified representation of a human lung
- Biological model. Time variation of $\mathrm{Na}^+, \mathrm{K}^+, \mathrm{Cl}^-$ ions represent synapse formation in a neuron
- Social model. A network is a representation of social structure

Definition. *A mathematical model is a quantitative statement of a model*

Is a quantitative statement of the above model examples possible?

- Physics model. Yes, $F = kx$, Hooke's law relating force F pulling on bar to elongation x.
  Note: In physics, a model with experimental support is called a *law*.
- Anatomy model. No.
- Biological model. Yes, Hodgkin-Huxley model.
- Social model. Yes, Six degrees of separation model.

Remark. A common scenario is that some states of a system are known along with a rate of change.

- If the system response is continuous, the system can be described through a *differential model*.
  - System responds to a single varying parameter: *ordinary differential equation model*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}) \,(\text{explicit form}), \boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = 0 \,(\text{implicit form})$$

  - System responds to multiple varying parameters: *partial differential equation model* $\partial_t u + c\partial_x u = 0$.
- If the system response is discontinuous, the system can be described through a *difference model*.

- Define key features of the system of interest: force F, elongation x for elastic bar model
- Formulate a mathematical statement relating identified features: $F = kx$, Hooke's law
- Define a model closure: from n observations $(x_i, F_i), i = 1, \ldots, n$ determine k by least squares
- Solve the model for cases of interest: for given force F^*, the model predicts elongation $x^* = F^* / k$
- Interpret model predictions (solutions): compare predicted x^* to observed (experimental) value x^{ex}

System	Model	Solution
Falling stone	$y''(x) = g$	$c_2 x + c_1 + \frac{g x^2}{2}$
Vibrating mass-spring	$k\, y(x) + m\, y''(x) = 0$	$c_2 \sin\left(\frac{\sqrt{k}\, x}{\sqrt{m}}\right) + c_1 \cos\left(\frac{\sqrt{k}\, x}{\sqrt{m}}\right)$
Beam deformation	$\mathrm{EI} y^{(4)}(x) = \sin(x)$	$c_4 x^3 + c_3 x^2 + c_2 x + c_1 + \frac{\sin(x)}{\mathrm{EI}}$
Parachutist	$m\, v'(x) = g\, m - b\, v(x)^2$	$\frac{\sqrt{g}\,\sqrt{m}\tanh\left(\frac{\sqrt{b}\, c_1 \sqrt{g}\, m + \sqrt{b}\,\sqrt{g}\, x}{\sqrt{m}}\right)}{\sqrt{b}}$

Table 1. Examples of physical models expressed through ODEs

Definition. *Solution of an ODE*
$h\colon (a,b) \to \mathbb{R}, h \in C^1[a,b]$ *is a* solution *of* $y' = f(x,y)$ *if* $h'(x) = f(x, h(x))\ \forall x \in (a,b)$

Remark.

- The above is a mathematical statement of the natural language "Concept of Solution", p. 4 of textbook
- $C^m[a,b]$ is the set of functions defined on $[a,b]$, with continuous derivatives up to and including order m
- $h'(x) = f(x, h(x))$ is a mathematical equation that evaluates to true or false
- $\forall$ means "for any"

Example 1. Solution verification

```
In[185]:= Eq = x y'[x] == -y[x]
```

$x\, y'(x) = -y(x)$

```
In[186]:= h[x_] = c/x
```

$\frac{c}{x}$

```
In[187]:= Eq /. y->h
```

True

```
In[188]:=
```

Definition. *If* $h(x)$ *is a solution of* $y' = f(x, y)$ *then the geometric locus* $\{(x, h(x)), x \in [a, b]\}$ *is a solution curve.*

Example 2. Solution curves

ODE

```
In[6]:= Eq = y'[x] == Cos[x]
```

$y'(x) = \cos(x)$

```
In[7]:=
```

General solution

```
In[7]:= Eqsol = DSolve[Eq,y[x],x][[1,1]] /. C[1]->c
```

$y(x) \to c + \sin(x)$

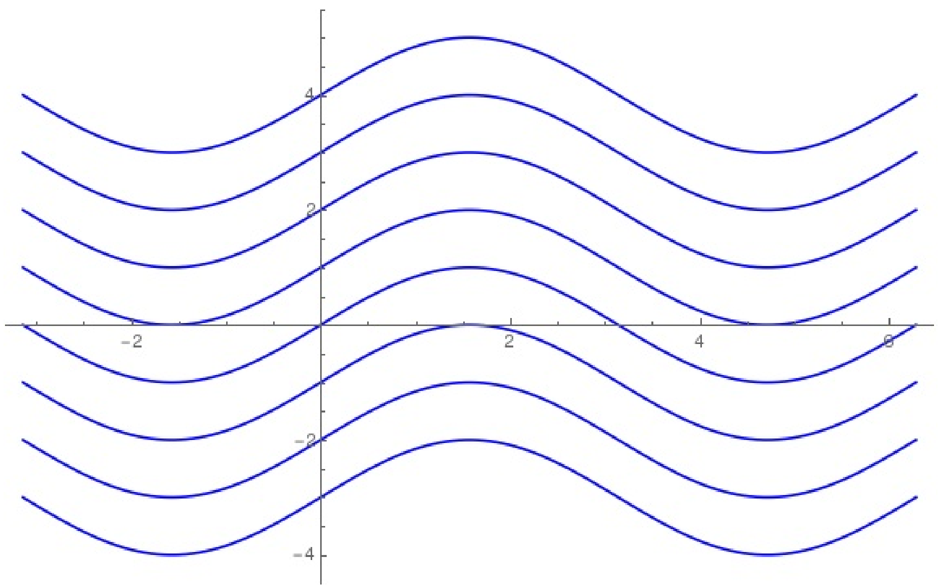
```
In[8]:=
```

Instances and plot file:

```
In[9]:= solutions = Table[y[x] /. Eqsol,{c,-3,4,1}];
        SolutionCurves = Plot[solutions,{x,-Pi,2Pi},PlotStyle->Blue,ImageSize->Large];
        Export["/home/student/courses/MATH528/L1Fig1.png",SolutionCurves]
```

/home/student/courses/MATH528/L1Fig1.png

```
In[10]:=
```



Example 3. Solution curves

ODE

```
In[14]:= Eq = y'[x] == 0.2 y[x]
```

$y'(x) = 0.2\, y(x)$

```
In[15]:=
```

General solution

```
In[15]:= Eqsol = DSolve[Eq,y[x],x][[1,1]] /. C[1]->c
```

$y(x) \to c\, e^{0.2\, x}$

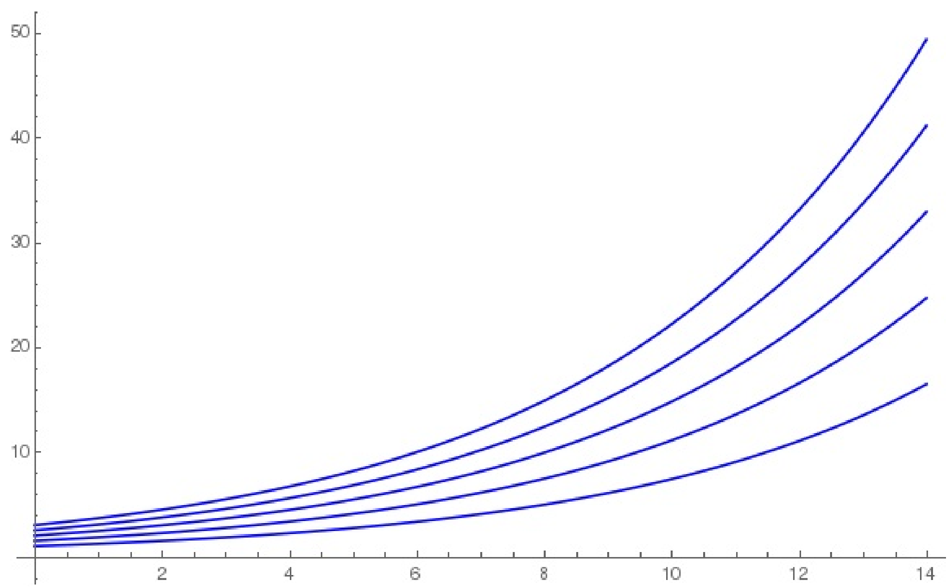
```
In[16]:=
```

Instances and plot file:

```
In[16]:= solutions = Table[y[x] /. Eqsol,{c,1,3,.5}];
         SolutionCurves = Plot[solutions,{x,0,14},PlotStyle->Blue,ImageSize->Large];
         Export["/home/student/courses/MATH528/L1Fig2.png",SolutionCurves]
```

/home/student/courses/MATH528/L1Fig2.png

```
In[17]:=
```



Example 4. Solution curves

ODE

```
In[18]:= Eq = y'[x] == -0.2 y[x]
```

$y'(x) = -0.2\, y(x)$

```
In[19]:=
```

General solution

```
In[19]:= Eqsol = DSolve[Eq,y[x],x][[1,1]] /. C[1]->c
```

$y(x) \to c\, e^{-0.2 x}$

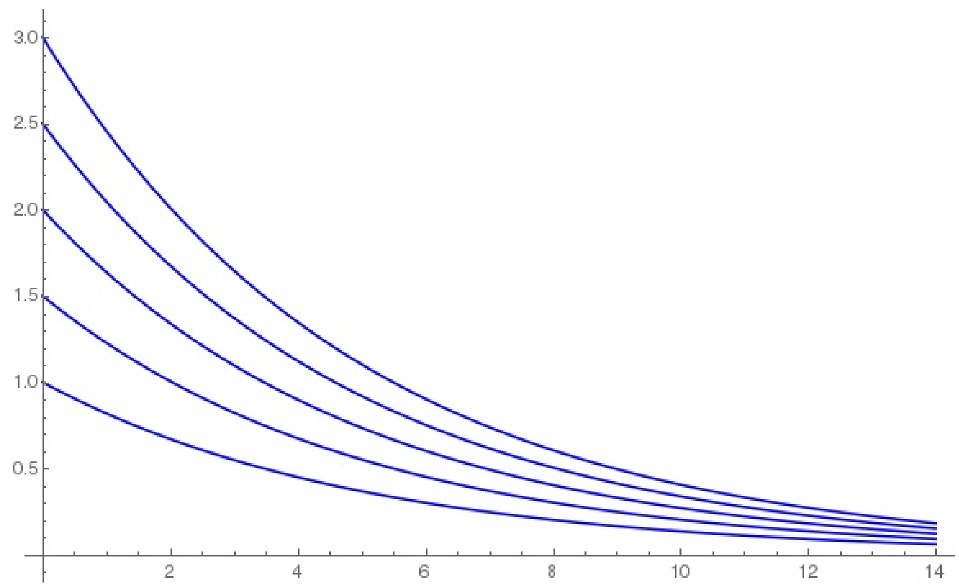
```
In[20]:=
```

Instances and plot file:

```
In[20]:= solutions = Table[y[x] /. Eqsol,{c,1,3,.5}];
         SolutionCurves = Plot[solutions,{x,0,14},PlotStyle->Blue,ImageSize->Large];
         Export["/home/student/courses/MATH528/L1Fig3.png",SolutionCurves]
```

/home/student/courses/MATH528/L1Fig3.png

```
In[21]:=
```



Example 5. The initial condition selects one member of the family of solutions

ODE and initial condition

```
In[194]:= Eq = {y'[x]==3y[x], y[0]==5.7}
```

$\{y'(x) = 3\,y(x), y(0) = 5.7\}$

```
In[195]:=
```

IVP solution

```
In[195]:= Eqsol = DSolve[Eq,y[x],x][[1,1]]
```

$y(x) \to 5.7\,e^{3x}$

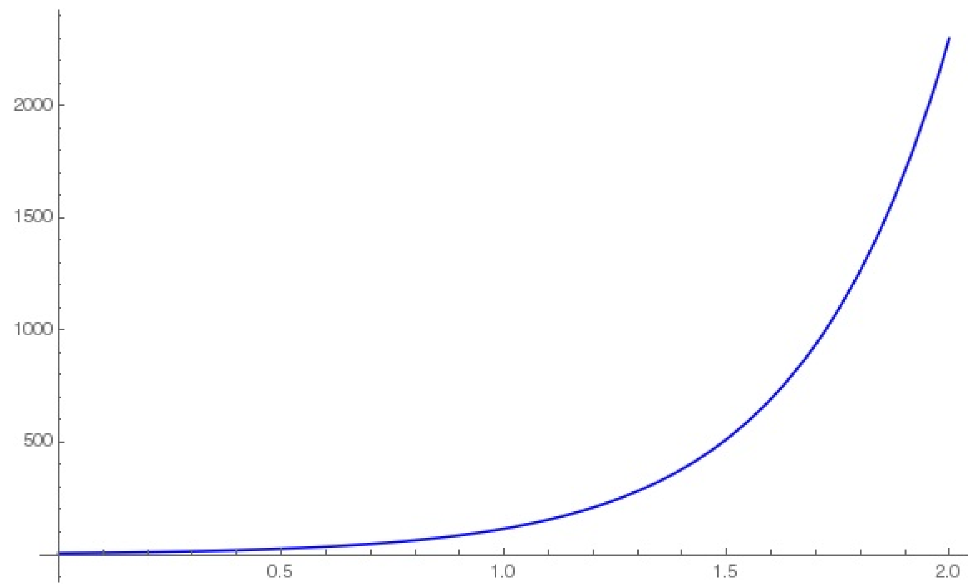
```
In[196]:=
```

Instances and plot file:

```
In[196]:= solution = y[x] /. Eqsol;
          SolutionCurve = Plot[solution,{x,0,2},PlotStyle->Blue,ImageSize->Large];
          Export["/home/student/courses/MATH528/L1Fig4.png",SolutionCurve]
```

/home/student/courses/MATH528/L1Fig4.png

```
In[197]:=
```



- The vector $\vec{\tau} = (1, f(x,y))$ is tangent to a solution of the ODE $y' = f(x,y)$

```
In[40]:= f[x_,y_] = x + y
```

$x + y$

```
In[41]:=
```

```
In[41]:= Eq = y'[x] == f[x,y[x]]
```

$y'(x) = y(x) + x$

```
In[42]:=
```

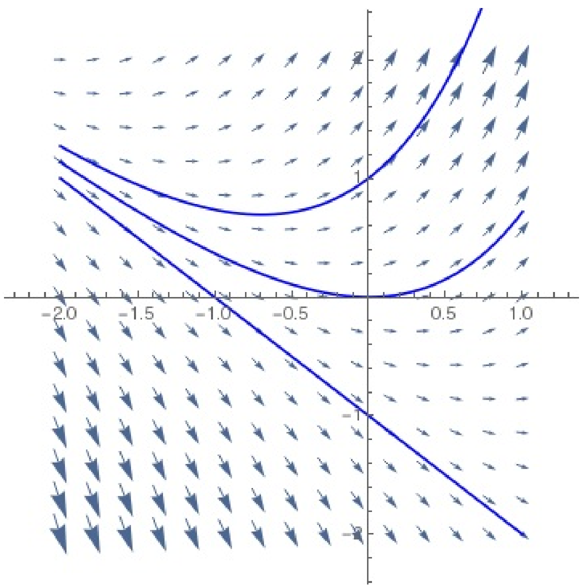
```
In[42]:= DirectionField=VectorPlot[{1,f[x,y]},{x,-2,1},{y,-2,2},Axes->True,Frame->False];
         Solutions = Table[y[x] /. DSolve[{Eq,y[0]==y0},y[x],x][[1,1]],{y0,-1,1,1}]
```

$\{-x-1, -x+e^x-1, -x+2\,e^x-1\}$

```
In[43]:= SolutionCurves = Plot[Solutions,{x,-2,1},ImageSize->Large,PlotStyle->Blue];
         SuperimposedPlots = Show[{DirectionField,SolutionCurves}];
         Export["/home/student/courses/MATH528/L1Fig5.png",SuperimposedPlots]
```

/home/student/courses/MATH528/L1Fig5.png

```
In[44]:=
```



Euler's numerical method for solving ODEs

- For $x_i = x_0 + ih$, construct an approximate solution $Y_i \cong y(x_i)$, by $Y_i = Y_{i-1} + h\,f(x_{i-1}, Y_{i-1})$
- To be completed in Lab01

Definition. *The ODE* $y' = h(x, y)$ *is separable if* $h(x, y)$ *can be expressed as* $h(x, y) = f(x)\,/\,g(y)$.

The solution of a separable equation is given implicitly by $F(x, y) = \int g(y)\,\mathrm{d}y - \int f(x)\,\mathrm{d}x + c$

Example. $y' = 1 + y^2$

```
In[201]:= h[x_,y_]=1+y^2
```

$y^2 + 1$

```
In[202]:=
```

```
In[202]:= f[x_]=1;
          g[y_]=1/(1+y^2);
          h[x,y]==f[x]/g[y]
```

True

```
In[203]:=
```

```
In[203]:= F[x_,y_,c_]=Integrate[g[y],y]-
          Integrate[f[x],x]+c
```

$c - x + \tan^{-1}(y)$

```
In[204]:=
```

Example. $y' = (1 + x)e^{-x}y^2$

```
In[204]:= h[x_,y_]=(1+x)Exp[-x]
          y^2
```

$e^{-x}\,(x + 1)\,y^2$

```
In[205]:=
```

```
In[205]:= f[x_]=(1+x)Exp[-x];
          g[y_]=1/y^2;
          h[x,y]==f[x]/g[y]
```

True

```
In[206]:=
```

```
In[206]:= F[x_,y_,
          c_]=Integrate[g[y],y]-
          Integrate[f[x],x]+c
```

$c - e^{-x}(-x - 2) - \frac{1}{y}$

```
In[207]:=
```