**Remark.** Many mathematical techniques involve *linear combinations*

- in linear algebra, $\alpha\boldsymbol{u}+\beta\boldsymbol{v}$, $\alpha,\beta\in\mathbb{C}$ scalars, $\boldsymbol{u},\boldsymbol{v}\in\mathbb{C}^n$, vectors
- in calculus, $\mathrm{d}(\alpha f+\beta g)=\alpha\mathrm{d}f+\beta\mathrm{d}g$, $\alpha,\beta\in\mathbb{R}$ scalars, $f,g:\mathbb{R}\to\mathbb{R}$, functions

**Definition.** *A first-order ODE of form $y'+p(x)y=r(x)$ is said to be linear.*

**Definition.** *A first-order ODE of form $y'+p(x)y=0$ is said to be homogeneous linear.*

**Remark.** A first-order homogeneous linear ODE is separable with solution

$$\frac{\mathrm{d}y}{y}=-p\mathrm{d}x\Rightarrow\ln y=-\int p(x)\mathrm{d}x+\ln c\Rightarrow y=ce^{-\int p(x)\mathrm{d}x}$$

**Remark.** A first-order non-homogeneous linear ODE leads to a differential form

$$(p(x)y-r(x))\,\mathrm{d}x+\mathrm{d}y=0, P(x,y)=p(x)\,y-r(x), Q(x,y)=1,$$

for which

$$\frac{1}{Q(x,y)}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=p(x),$$

such that

$$F(x)=\exp\int p(x)\,\mathrm{d}x=\exp h,$$

is an integrating factor with property $F'=pF$ such that the linear ODE can be rewritten in separable form

$$Fy'+Fpy=Fy'+F'\,y=\frac{\mathrm{d}}{\mathrm{d}x}(Fy)=r$$

**Corollary.** *(of Theorem 1, Lesson 2) The solution to* $y' + p(x)y = r(x)$ *is*

$$y(x) = e^{-h(x)}\left(\int e^{h(x)}\, r(x)\, \mathrm{d}x + c\right), h(x) = \int p(x)\, \mathrm{d}x$$

```
In[10]:= DSolve[y'[x] + p[x] y[x] == r[x], y[x], x]
```

$$\left\{\left\{y(x) \to c_1\, e^{\int_1^x -p(K[1])\, dK[1]} + e^{\int_1^x -p(K[1])\, dK[1]} \int_1^x r(K[2])\, e^{-\int_1^{K[2]} -p(K[1])\, dK[1]}\, dK[2]\right\}\right\}$$

```
In[11]:= h[x_] = Integrate[p[x],x]
```

$$\int p(x)\, dx$$

```
In[13]:= y[x_] = Exp[-h[x]] ( Integrate[ Exp[h[x]] r[x], x ] + c)
```

$$e^{-\int p(x)\, dx}\left(c + \int r(x)\, e^{\int p(x)\, dx}\, dx\right)$$

```
In[14]:= linODE = y'[x] + p[x] y[x] == r[x]
```

*True*

```
In[15]:=
```

**Example 1.** $y' + y\tan x = \sin 2x$, $y(0) = 1$.

```
In[15]:= y[x]
```

$$e^{-\int p(x)\,dx}\left(c + \int r(x)\,e^{\int p(x)\,dx}\,dx\right)$$

```
In[19]:= sol[x_]=y[x] /. {p[x]->Tan[x], r[x]->Sin[2x]}
```

$$\cos(x)\,(c - 2\cos(x))$$

```
In[21]:= Simplify[sol'[x] + sol[x] Tan[x] == Sin[2x]]
```

True

```
In[23]:= Simplify[DSolve[z'[x] + Tan[x] z[x] == Sin[2x], z[x], x]]
```

$$\{\{z(x) \to \cos(x)\,(c_1 - 2\cos(x))\}\}$$

```
In[24]:=
```

# Linear ODE example: RL electric circuit, electro-mechanic analogy

| Element | Electrical | Mechanical |
|---|---|---|
| Independent variable | time $t$ | time $t$ |
| Dependent variable | charge $q(t)$ | position $x(t)$ |
| Rate of change of dependent variable | current $i = q'(t)$ | velocity $v(t) = x'(t)$ |
| Inertia | inductance $L$ | mass $m$ |
| Dissipation | resistor $R$ | drag $b$ |
| Storage | capacitance $C$ | spring $k$ |
| Forcing | external voltage $u(t)$ | external force $f(t)$ |
| Inertia-dissipation model | $Li' + Ri = u$ | $mv' + bv = f$ |

```
In[36]:= sol[t_] = Apart[i[t] /. DSolve[ {L i’[t] + R i[t] == u, i[0]==j}, i[t], t][[1,1]]];
         plt = Plot[ Table[sol[t] /. {R->12, L->0.1, u->48}, {j,0,8,2}],{t,0,0.05},PlotRange->All];
         Export["/home/student/courses/MATH528/L03Fig01.pdf",plt]; sol[t]
```

$$\frac{(j\,R-u)\,e^{-\frac{R\,t}{L}}}{R}+\frac{u}{R}$$

```
In[37]:=
```

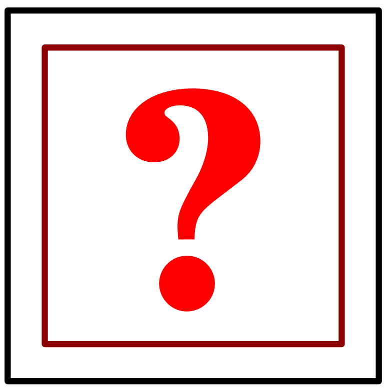

**Figure 1.** Inertia-dissipation model returns to steady-state after initial perturbation

**Definition.** *A non-linear first-order ODE of form $y' + p(x)y = g(x)\, y^a$ is said to be a Bernoulli equation.*

**Remark.** A Bernoulli equation can be brought to linear form through the substitution $u(x) = [y(x)]^{1-a}$

```
In[34]:= Clear[y]; BernoulliEq = y'[x] + p[x] y[x] == g[x] y[x]^a
```

$$p(x)\, y(x) + y'(x) = g(x)\, y(x)^a$$

```
In[32]:= y[x_] = u[x]^(1/(1-a))
```

$$u(x)^{\frac{1}{1-a}}$$

```
In[33]:= TransformedEq = FullSimplify[BernoulliEq]
```

$$\frac{u(x)^{-\frac{a}{a-1}}\left((a-1)\, p(x)\, u(x) - u'(x)\right)}{a-1} = g(x)\left(u(x)^{\frac{1}{1-a}}\right)^a$$

**Example.** Logistic equation $y' = ay - by^2$, $u = y^{1-a} = y^{-1}$, $u' + au = b$