Example 1. The IVP $F(x,y,y')=|y'|+|y|=0$, $y(0)=1$ has no (*zero*) solutions.

Example 2. The IVP $y'=f(x,y)=2x$, $y(0)=c$ has *one* solution, $y(x)=x^2+c$ for any given c.

Example 3. The IVP $y'=f(x,y)=ny^{(n-1)/n}$, $n\in\{2,3,\ldots\}$, $y(0)=0$ has *two* solutions:

1. $y(x)=0$, and
2. $y(x)=x^n$

Example 4. The IVP $y'=(y-1)/x$, $y(0)=1$ has *infinitely* many solutions, $y=1+cx$.

⇑ Basic questions: existence and uniqueness of solutions to IVP. First, investigate the direction fields of the above examples.

```
In[4]:= f = {f1a,f1b,f2,f3,f4} = {y, -y, 2x, 3 y^(2/3), (y-1)/x};
        vPlt1a = VectorPlot[{1,f1a},{x,-1,1},{y,-0.5,1.5}];
        vPlt1b = VectorPlot[{1,f1b},{x,-1,1},{y,-0.5,1.5},VectorStyle->Red];
        vPlt1=Show[{vPlt1a,vPlt1b}];
        vPlt = Prepend[ Map[VectorPlot[{1,#},{x,-1,1},{y,-0.5,1.5}]&,f[[3;;]]], vPlt1 ];
        plots = GraphicsGrid[ ArrayReshape[vPlt,{2,2}] ];
        Export["/home/student/courses/MATH528/L04Fig01.pdf",plots]
```

/home/student/courses/MATH528/L04Fig01.pdf

```
In[5]:=
```

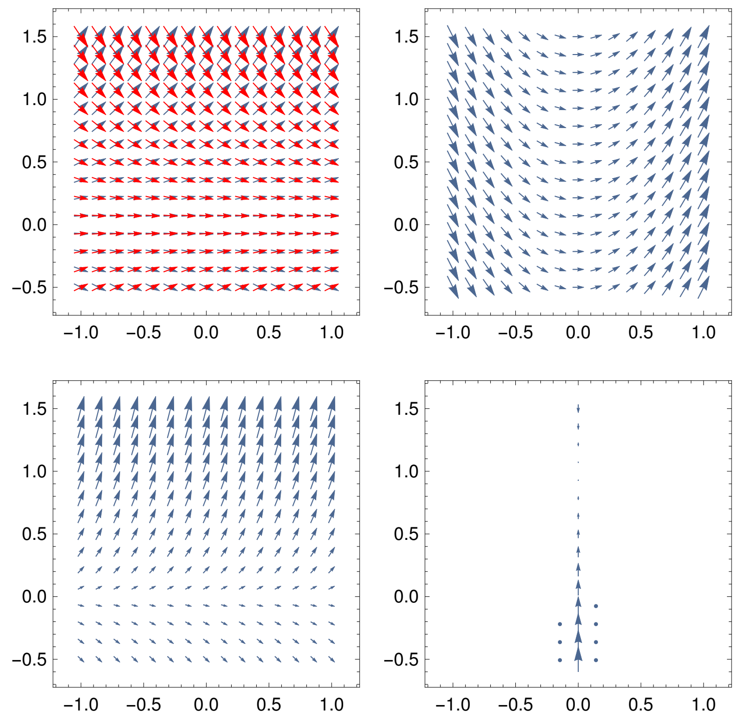

Figure 1. Example direction fields

Definition. *$f: R \subseteq \mathbb{R}^2$ is continuous at $(x_0, y_0) \in R$ if $\forall \varepsilon > 0$, $\exists \delta_\varepsilon$ such that $|x - x_0| + |y - y_0| < \delta_\varepsilon$ implies*

$$|f(x, y) - f(x_0, y_0)| < \varepsilon.$$

Definition. *$f: R \subseteq \mathbb{R}^2$ is bounded over R if $\exists K \in \mathbb{R}$, finite such that $|f(x, y)| \leqslant K$, $\forall (x, y) \in R$.*

Theorem. *A solution exists to the IVP $y' = f(x, y), y(x_0) = y_0$ if f is continuous and bounded in some rectangle $R: |x - x_0| < a, |y - y_0| < b$.*

Definition. *$f: R \subseteq \mathbb{R}^2$ is Lipshitz continuous (in y) over R if $\forall (x, y_1), (x, y_2) \in R$, $\exists M \in \mathbb{R}$ finite such that*

$$|f(x, y_2) - f(x, y_1)| \leqslant M |y_2 - y_1|.$$

Theorem. *A unique solution exists to the IVP $y' = f(x, y), y(x_0) = y_0$ if f is continuous and bounded in x, and Lipschitz continuous and bounded in y over some rectangle $R: |x - x_0| < a, |y - y_0| < b$.*

Example. For the IVP $y' = f(x, y) = n y^{(n-1)/n}, n \in \{2, 3, \ldots\}, y(0) = y_0 = 0$,

$$\frac{|f(x, y_1) - f(x, y_0)|}{|y_1 - y_0|} = \frac{n y_1^{(n-1)/n}}{y_1} = \frac{n}{y_1^{1/n}}, \text{arbitrarily large as } y_1 \to 0.$$

- Write solution to IVP $y' = f(x,y),\, y(x_0) = y_0$ as

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, \mathrm{d}t$$

- Above formula suggests constructing a sequence of approximations $\{y_1(x), y_2(x), \ldots, y_n(x), y_{n+1}(x), \ldots\}$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, \mathrm{d}t, n = 0, 1, 2, \ldots$$

Example. Consider $y' = x + y, y(0) = 0$. Choose initial approximation $y_0(x) = 0$. Compute:

```
In[15]:= Clear[f]; f[x_,y_]=x+y; ODE = y'[x] == f[x,y[x]]; IniCond = y'[0] == 0;
         sol[x_] = y[x] /. DSolve[{ODE,IniCond},y[x],x][[1,1]];
         {sol[x], Normal[Series[sol[x],{x,0,4}]]} // TableForm
```

$$-x + e^x - 1$$
$$\frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2}$$

```
In[16]:= Clear[y]; y[0,x_]=0; y[n_,x_] := y[n,x] = y[0,0] + Integrate[f[t,y[n-1,t]],{t,0,x}];
         Table[y[n,x],{n,1,3}] // TableForm
```

$$\frac{x^2}{2}$$
$$\frac{x^3}{6} + \frac{x^2}{2}$$
$$\frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2}$$

```
In[17]:=
```

```
In[19]:= solPlot = Plot[sol[x],{x,0,3},PlotStyle->Red];
         yPlots = Plot[Table[y[n,x],{n,1,5}],{x,0,3}];
         PicardPlots = Show[Flatten[{solPlot,yPlots}]];
         Export["/home/student/courses/MATH528/L04Fig02.pdf",PicardPlots]
```

/home/student/courses/MATH528/L04Fig02.pdf

```
In[20]:=
```

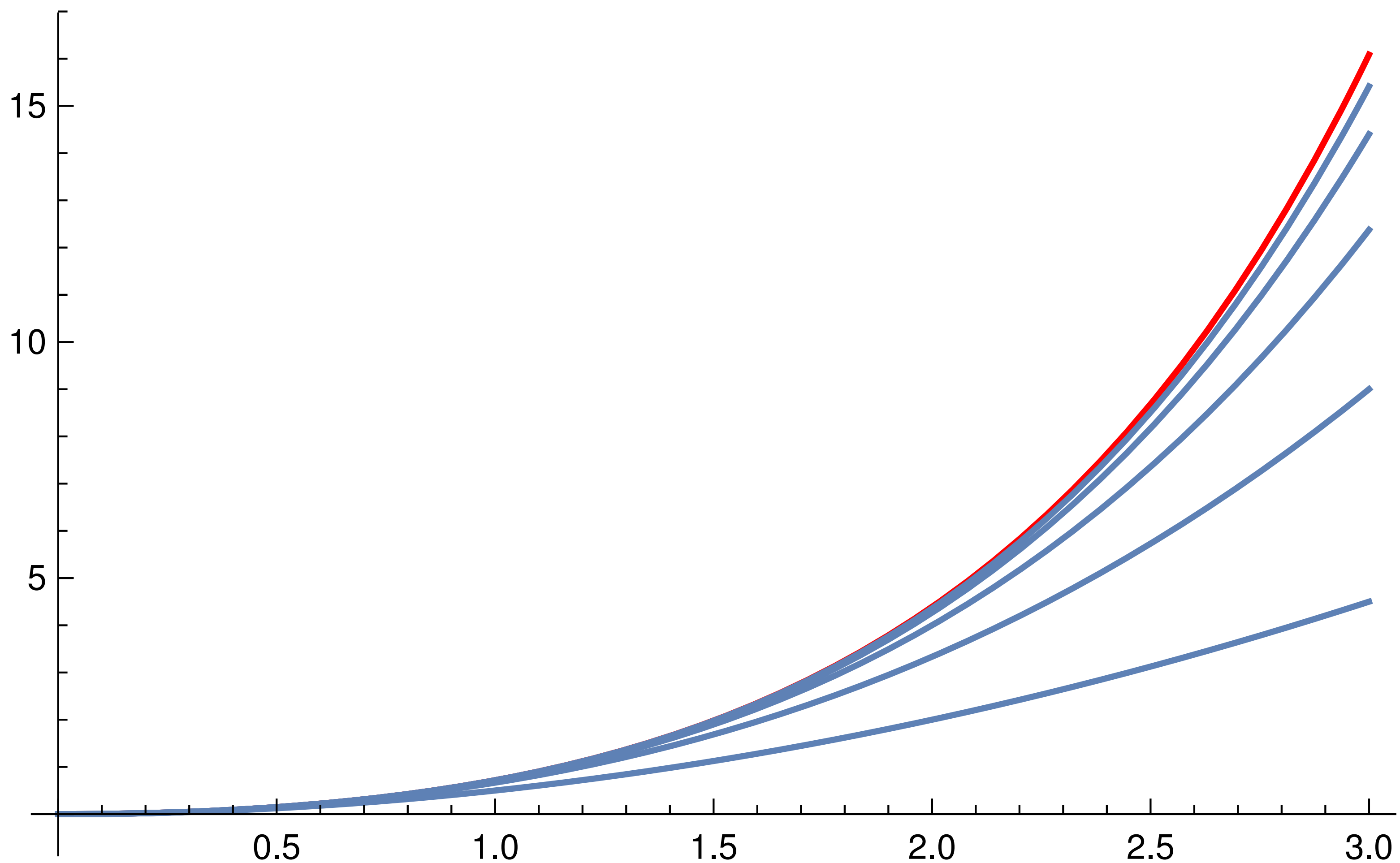

Figure 2. Sequence of Picard approximants to exact solution

Definition. $y'' + p(x)y' + q(x)y = r(x)$ *is a linear second-order ODE.*

Definition. $y'' + p(x)y' + q(x)y = 0$ *is a homogeneous, linear second-order ODE.*

Remark. If y_1, y_2 are solutions of a homogeneous linear ODE, the $c_1y_1 + c_2y_2$ is a solution.

Definition. $y_1(x), y_2(x)$ *are linearly independent if* $c_1y_1(x) + c_2y_2(x) = 0$ *implies* $c_1 = c_2 = 0$.

Definition. *Independent solutions* y_1, y_2 *of* $y'' + p(x)y' + q(x)y = 0$ *form a basis or fundamental system.*

Definition. *If* y_1, y_2 *are independent solutions of* $y'' + p(x)y' + q(x)y = 0$, $c_1y_1 + c_2y_2$ *for arbitrary* $c_1, c_2 \in \mathbb{R}$ *is called the general solution, and the particular solution for specific values* c_1, c_2.

Algorithm Reduction of order

If $y_1(x)$ is a solution of $y'' + p(x)y' + q(x)y = 0$, the substitution $y(x) = u(x)y_1(x)$ leads to the first order ODE

$$y_1 v' + (2y_1' + py_1)v = 0$$

with $v(x) = u'(x)$.