Definition. *Equations of the form* $x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} \ldots + a_1 x y' + a_0 y = r(x)$ *are called Euler-Cauchy equations.*

Solution of homogeneous equation

- Guess solution is of form $y(x) = x^m$, $y' = mx^{m-1}$, ..., $y^{(n)} = m(m-1)\ldots(m-n+1)x^{m-n}$ and obtain

$$m(m-1)\ldots(m-n+1) + a_{n-1}m(m-1)\ldots(m-n+2) + \cdots + a_1 m + a_0 = 0,$$

  a polynomial equation in m.
- For $n = 2$, with notation $a_1 = a$, $a_0 = b$

$$m(m-1) + am + b = m^2 + (a-1)m + b = 0$$

Note. A constant-coefficient equation, $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 = 0$, is a linear combination of operators $\frac{\mathrm{d}^k}{\mathrm{d}x^k}$ for which e^{rx} is an eigenfunction with eigenvalue r^k.

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(e^{rx}) = r^k e^{rx}$$

The homogeneous Euler-Cauchy equation is a linear combination of operators $x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}$ for which x^m is an eigenfunction with eigenvalue $m(m-1)\ldots(m-k+1)$

$$x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^m) = m(m-1)\ldots(m-k+1)x^m$$

Cases: $m^2+(a-1)m+b=0$ has

- distinct roots m_1, m_2: general solution is $y(x)=c_1x^{m_1}+c_2x^{m_2}$
- double root m. Apply reduction of order technique. Seek solution of form $y(x)=u(x)\,y_1(x)=u(x)\,x^m$

$$\begin{aligned}
x^2y''+axy'+by &= x^2(u''y_1+2u'y_1'+uy_1'')+ax(u'\,y_1+uy_1')+buy_1\\
&= x^2y_1v'+(2x^2y_1'+axy_1)v+(x^2y_1''+axy_1'+by_1)u=0\Rightarrow\\
x^{m+2}v'+(2m+a)x^{m+1}v &= 0,\text{ with } v=u'\Leftrightarrow u(x)=\int v(x)\mathrm{d}x\\
xv'+(2m+a)v &= 0\Rightarrow xv'(x)+v(x)=0\Rightarrow v(x)=1/x\Rightarrow u(x)=\ln x\\
y(x) &= (c_1+c_2\ln x)x^m
\end{aligned}$$

```
In[12]:= Clear[y]; y[x_] = Integrate[v[x],x] x^m;
         rhs = Simplify[x^2 y''[x] + a x y'[x] + b y[x] /. b -> (-m^2-(a-1)m) /. m->(1-a)/2]
```

$x^{\frac{3}{2}-\frac{a}{2}}(x\,v'(x)+v(x))$

```
In[15]:= ODE = rhs / x^(3/2-a/2) == 0
```

$x\,v'(x)+v(x)=0$

```
In[19]:= u[x_] = Integrate[v[x] /. DSolve[ODE,v[x],x][[1,1]],x]
```

$c_1\log(x)$

```
In[20]:=
```

Definition. *The Wronskian of functions* $\{y_1(x), ..., y_n(x)\}$ *is the determinant*

$$W(y_1, ..., y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Theorem. *Solutions* y_1, y_2 *of the ODE* $y'' + p(x)\, y' + q(x) = 0$, p, q *continuous on* $I = (a, b)$ *are linearly dependent on* I *iff* $W(y_1, y_2) = 0$ *at some* $x_0 \in I$. *Furthermore if* $W = 0$ *at some* x_0 *then* $W = 0$ *on* I.

Definition. *A general solution $y(x)$ of the inhomogeneous ODE $L[y] = y'' + p(x)\,y' + q(x)\,y = r(x)$ is the sum*

$$y(x) = y_h(x) + y_p(x), \text{with } L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)$$

of the general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous form of the ODE ($y'' + p(x)\,y' + q(x)\,y = 0$) and a particular solution of the inhomogeneous form, $L[y] = L[y_h + y_p] = L[y_p] = r$.

Algorithm : Method of undetermined coefficients

For the constant-coefficient $y'' + ay' + by = r(x)$, seek a particular solution as a linear combination of $r(x)$ and its derivatives

Example. $y'' + y = 0.001x^2, y(0) = 0, y'(0) = 1.5$

```
In[29]:= Clear[y]; rhs = y''[x] + y[x]; r[x_] = x^2/1000;
         hODE = rhs == 0; ODE = rhs == r[x]; iCond1 = y[0]==0; iCond2 = y'[0]==3/2;
         hsol[x_] = y[x] /. DSolve[hODE,y[x],x][[1,1]];
         gsol[x_] = y[x] /. DSolve[ODE,y[x],x][[1,1]];
         sol[x_] = y[x] /. DSolve[{ODE,iCond1,iCond2},y[x],x][[1,1]];
         {hsol[x],gsol[x],sol[x]}
```

$$\left\{c_2 \sin(x) + c_1 \cos(x), c_2 \sin(x) + c_1 \cos(x) + \frac{x^2 - 2}{1000}, 0.001\,x^2 + 1.5 \sin(1.\,x) + 0.002 \cos(1.\,x) - 0.002\right\}$$

```
In[25]:=
```