**Definition.** *The general solution of the homogeneous ODE*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{1}$$

*is* $y(x) = c_1y_1(x) + \cdots + c_ny_n(x)$ *with* $\{y_1(x), \ldots, y_n(x)\}$ *linearly independent.*

**Definition.** *An initial value problem is defined by (1) and* $n$ *initial conditions*

$$y(x_0) = K_0, y'(x_0) = K_1, \ldots, y^{(n-1)}(x_0) = K_{n-1}. \tag{2}$$

**Theorem.** *If* $p_i(x)$ *are continuous on* $I = (a, b)$, *and* $x_0 \in I$, *then the IVP (1-2) has an unique solution on I.*

**Theorem.** $\{y_1(x), \ldots, y_n(x)\}$ *are linearly dependent iff* $\exists x_0$ *such that*

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0$$

```
In[1]:= Wronskian[{1,x,x^2,x^3},x]
```

12

```
In[2]:= Wronskian[{Cos[x], Sin[x], x Cos[x], x Sin[x]},x]
```

4

```
In[3]:=
```

The characteristic polynomial of the constant-coefficient ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

is

$$P_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

- If $P_n(\lambda)$ has distinct roots, $W(e^{\lambda_1 x}, \ldots, e^{\lambda_n x}) \neq 0$, and $y_i = e^{\lambda_i x}$ are linearly independent
- For any complex root $\lambda = \gamma + i\omega$, basis functions are $\{e^{\gamma x}\cos(\omega x), e^{\gamma x}\sin(\omega x)\}$
- For any $m$-repeated root $m$, $\{e^{\lambda x}, x e^{\lambda x}, \ldots, x^{m-1}e^{\lambda x}\}$ are linearly independent.

```
In[1]:= ODE = D[y[x],{x,5}] - 5 D[y[x],{x,3}] + 4 y'[x] == 0
```

$$y^{(5)}(x) - 5\,y^{(3)}(x) + 4\,y'(x) = 0$$

```
In[2]:= iCond = {y'[0]==-5, y''[0]==11, y'''[0]==-23, Evaluate[D[y[x],{x,4}] /. x->0] == 47 }
```

$$\{y'(0) = -5, y''(0) = 11, y^{(3)}(0) = -23, y^{(4)}(0) = 47\}$$

```
In[3]:= DSolve[Flatten[{ODE,iCond}],y[x],x]
```

$$\{\{y(x) \to e^{-2x}\,(c_5\,e^{2x} - e^x + 3)\}\}$$

```
In[4]:=
```

- $y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = r(x)$, has solution $y(x) = y_h(x) + y_p(x)$ with:
  - $y_h(x)$ the general solution of the homogeneous ODE
  - $y_p(x)$ is a particular solution of the inhomogeneous ODE

Particular solutions are found by:

- method of undetermined coefficients
- method of variation of parameters

```
In[4]:= rhs = x^3 y'''[x] - 3 x^2 y''[x] + 6 x y'[x] - 6 y[x]
```

$$x^3\, y^{(3)}(x) - 3\, x^2\, y''(x) + 6\, x\, y'(x) - 6\, y(x)$$

```
In[5]:= lhs = x^4 Log[x]
```

$$x^4 \log(x)$$

```
In[6]:= DSolve[{rhs==lhs},y[x],x]
```

$$\left\{\left\{y(x) \to c_3\, x^3 + c_2\, x^2 + c_1\, x + \frac{1}{36}\left(6\, x^4 \log(x) - 11\, x^4\right)\right\}\right\}$$

```
In[7]:= DSolve[{rhs==0},y[x],x]
```

$$\{\{y(x) \to c_3\, x^3 + c_2\, x^2 + c_1\, x\}\}$$

```
In[8]:=
```