Definition. *The implicit form of a first-order ODE system is*

$$\boldsymbol{F}(t, \boldsymbol{y}, \boldsymbol{y}') = \boldsymbol{0}, \text{with } \boldsymbol{F}: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \boldsymbol{y}: \mathbb{R} \to \mathbb{R}^n.$$

Definition. *The explicit form of a first-order ODE system is*

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \text{with } \boldsymbol{f}: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \boldsymbol{y}: \mathbb{R} \to \mathbb{R}^n.$$

Definition. *A linear system of first-order ODEs has form*

$$\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y} + \boldsymbol{g}(t), \boldsymbol{A}: \mathbb{R} \to \mathbb{R}^{n \times n}, \boldsymbol{g}: \mathbb{R} \to \mathbb{R}^n.$$

Remark. An n^{th}-order ODE

$$u^{(n)} = g(t, u, u', \ldots, u^{(n-1)})$$

can be rewritten as a system of first-order ODEs

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}) = \begin{pmatrix} u \\ y_1' \\ \vdots \\ y_{n-2}' \\ g(t, \boldsymbol{y}) \end{pmatrix}.$$

Definition. *The initial value problem for a first-order ODE system is*

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \text{with } \boldsymbol{f}: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \boldsymbol{y}: \mathbb{R} \to \mathbb{R}^n, \boldsymbol{y}(t_0) = \boldsymbol{K} \in \mathbb{R}^n.$$

Theorem. *If* $\boldsymbol{f} \in C^1(R)$, *then the initial value problem for* $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$, with $\boldsymbol{f}: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{y}: \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{y}(t_0) = \boldsymbol{K} \in \mathbb{R}^n$, *has a unique solution over some subset of* R.

Definition. *A fundamental solution (basis) of the ODE system* $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ *is an independent set of* n *solutions*

$$\{\boldsymbol{y}^{(1)}(t), \ldots, \boldsymbol{y}^{(n)}(t)\},$$

that can be written in matrix form as $\boldsymbol{Y}(t) = \begin{pmatrix} \boldsymbol{y}^{(1)}(t) & \ldots & \boldsymbol{y}^{(n)}(t) \end{pmatrix}$

Theorem. *Solutions* $\boldsymbol{Y}(t)$ *constitute a fundamental solution if the wronskian* $W = \det \boldsymbol{Y} \neq 0$ *for any* t.

- A solution of the homogeneous equation is written as a linear combination of the basis, $\boldsymbol{y} = \boldsymbol{Y}\boldsymbol{c}$, with $\boldsymbol{c} \in \mathbb{R}^n$.

```
In[34]:= ODE = {u'[t] == 2 v[t], v'[t] == 8 u[t]};
         y[t_] = {u[t],v[t]} /. DSolve[ODE,{u[t],v[t]},t][[1]]
```

$$\left\{\frac{1}{2} c_1 e^{-4t} (e^{8t} + 1) + \frac{1}{4} c_2 e^{-4t} (e^{8t} - 1), c_1 e^{-4t} (e^{8t} - 1) + \frac{1}{2} c_2 e^{-4t} (e^{8t} + 1)\right\}$$

```
In[31]:= Y = Transpose[Map[Coefficient[y[t],C[#]]&,{1,2}]]; MatrixForm[Y]
```

$$\begin{pmatrix} \frac{1}{2} e^{-4t} (e^{8t} + 1) & \frac{1}{4} e^{-4t} (e^{8t} - 1) \\ e^{-4t} (e^{8t} - 1) & \frac{1}{2} e^{-4t} (e^{8t} + 1) \end{pmatrix}$$

```
In[33]:= Y . {C[1],C[2]} == y[t]
```

True

```
In[34]:=
```

Example: Car suspension

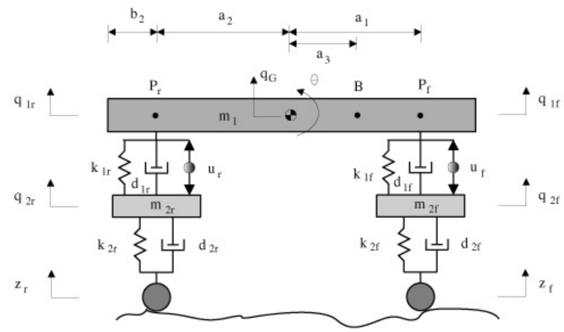

Figure 1. Half-vehicle model with rigid body.

- Terrain modeled by known functions $z_f(t), z_r(t)$
- Car suspension described by: $q_{1f}(t), q_{2f}(t), q_{1r}(t), q_{2r}(t)$
- Dynamics described by Newton's law for linear and angular momentum (assume $a_1 = a_2 = a$)

$$\begin{aligned}
m_{2r}\,\ddot{q}_{2r} &= k_{1r}(q_{1r} - q_{2r}) - k_{2r}(q_{2r} - z_r) + d_{1r}(\dot{q}_{1r} - \dot{q}_{2r}) - d_{2r}(\dot{q}_{2r} - \dot{z}_r) \\
m_{2f}\,\ddot{q}_{2f} &= k_{1f}(q_{1f} - q_{2f}) - k_{2f}(q_{2f} - z_f) + d_{1f}(\dot{q}_{1f} - \dot{q}_{2f}) - d_{2f}(\dot{q}_{2f} - \dot{z}_f) \\
m_1 \frac{\ddot{q}_{1r} + \ddot{q}_{1f}}{2} &= -k_{1r}(q_{1r} - q_{2r}) - k_{1f}(q_{1f} - q_{2f}) - d_{1r}(\dot{q}_{1r} - \dot{q}_{2r}) - d_{1f}(\dot{q}_{1f} - \dot{q}_{2f}) \\
I \frac{\ddot{q}_{1f} - \ddot{q}_{1r}}{2a} &= a[k_{1r}(q_{1r} - q_{2r}) + d_{1r}(\dot{q}_{1r} - \dot{q}_{2r})] - a[k_{1f}(q_{1f} - q_{2f}) + d_{1f}(\dot{q}_{1f} - \dot{q}_{2f})]
\end{aligned}$$