- Recall the solution to the IVP $y' = \lambda y$, $y(0) = 0$ is $y(t) = e^{\lambda t} y_0$, with $y: \mathbb{R} \to \mathbb{R}$
- The solution to the system IVP $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is similar, $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0$, with:
  - $\boldsymbol{y}_0 \in \mathbb{R}^n$, $\boldsymbol{y}: \mathbb{R} \to \mathbb{R}^n$
  - $\boldsymbol{A} \in \mathbb{R}^{n \times n}$
  - $e^{\boldsymbol{A}t} \in \mathbb{R}^{n \times n}$ is known as the matrix exponential defined by

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}\frac{t}{1!} + \boldsymbol{A}^2\frac{t^2}{2!} + \cdots + \boldsymbol{A}^k\frac{t^k}{k!} + \cdots$$

- If $\boldsymbol{A}$ is diagonalizable, i.e., the eigenvector matrix $\boldsymbol{X} \in \mathbb{R}^{n \times n}$

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \ldots & \boldsymbol{x}_n \end{pmatrix}, \boldsymbol{x}_k \in \mathbb{R}^n$$

  arising in the eigenvalue problem $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$ (or $\boldsymbol{A}\boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k$, $k = 1, \ldots, n$) is invertible, then

$$e^{\boldsymbol{A}t} = \boldsymbol{X}e^{\boldsymbol{\Lambda}t}\boldsymbol{X}^{-1}, e^{\boldsymbol{\Lambda}t} = \operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}), \boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

- Diagonalizable matrices include:
  - matrices with distinct eigenvalues, $j \neq k \Rightarrow \lambda_j \neq \lambda_k$
  - symmetric matrices, $\boldsymbol{A}^T = \boldsymbol{A}$, the eigenvalues of which are purely real
  - skew-symmetric matrices, $\boldsymbol{A}^T = -\boldsymbol{A}$, the eigenvalues of which are purely imaginary
- Assume henceforth that $\boldsymbol{A}$ is diagonalizable
- Rewrite solution as

$$\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0 = \boldsymbol{X}e^{\boldsymbol{\Lambda}t}\boldsymbol{X}^{-1}\boldsymbol{y}_0 = \boldsymbol{X}e^{\boldsymbol{\Lambda}t}\boldsymbol{c} = \begin{pmatrix} e^{\lambda_1 t}\boldsymbol{x}_1 & \ldots & e^{\lambda_n t}\boldsymbol{x}_n \end{pmatrix}\boldsymbol{c} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n$$

  and observe that $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\} = \{e^{\lambda_1 t}\boldsymbol{x}_1, \ldots, e^{\lambda_n t}\boldsymbol{x}_n\}$ are independent and form a basis for solutions of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$

- From $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0 = \boldsymbol{X}e^{\boldsymbol{\Lambda}t}\boldsymbol{c} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n$ and $\lambda_k = \alpha_k + i\beta_k$ observe:
  - If for any k, $\mathrm{Re}(\lambda_k) > 0$ and $c_k \neq 0$ then $\lim_{t\to\infty} \|\boldsymbol{y}(t)\| = \infty$
  - If for all k, $\mathrm{Re}(\lambda_k) < 0$, then $\lim_{t\to\infty} \|\boldsymbol{y}(t)\| = 0$
- In general, many qualitative features of the solution are discernible from the eigenvalues/eigenvectors

Definition. *The parametric curve* $\{y_1(t), \ldots, y_n(t)\}$ *is a trajectory in the phase plane* $y_1, \ldots, y_n$. *The set of all trajectories is the phase portrait of the ODE system.*

Example 1 (Improper node). $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ with

$$\boldsymbol{A} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

```
In[64]:= A={{-3,1},{1,-3}}; lambda=Eigenvalues[A]; Lambda=DiagonalMatrix[lambda];
         X=Transpose[Eigenvectors[A]]; Y[t_] = X . Exp[Lambda t]; Map[MatrixForm[#]&,{Lambda,X,Exp[Lambda
         t],Y[t]}]
```

$$\left\{ \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} e^{-4t} & 1 \\ 1 & e^{-2t} \end{pmatrix}, \begin{pmatrix} 1-e^{-4t} & -1+e^{-2t} \\ 1+e^{-4t} & 1+e^{-2t} \end{pmatrix} \right\}$$

```
In[65]:= A . X == X . Lambda
```

True

```
In[66]:= y[t_,c_] := Y[t] . c; {y[t,{1,0}],y[t,{0,1}]}
```

$$\begin{pmatrix} 1-e^{-4t} & 1+e^{-4t} \\ -1+e^{-2t} & 1+e^{-2t} \end{pmatrix}$$

$$\boldsymbol{A} \in \mathbb{R}^{2\times 2}, \boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0 = \boldsymbol{X}e^{\boldsymbol{\Lambda}t}\boldsymbol{X}^{-1}\boldsymbol{y}_0 = \boldsymbol{X}e^{\boldsymbol{\Lambda}t}\boldsymbol{c} = \begin{pmatrix} e^{\lambda_1 t}\boldsymbol{x}_1 & e^{\lambda_2 t}\boldsymbol{x}_2 \end{pmatrix}\boldsymbol{c} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$$

```
In[21]:= PhasePortrait = StreamPlot[A.{y1,y2},{y1,-1,1},{y2,-1,1},StreamColorFunction->Hue,
           FrameLabel->{Subscript["y",1],Subscript["y",2]},PlotLegends->Automatic];
         Export["/home/student/courses/MATH528/L10Fig01.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L10Fig01.pdf

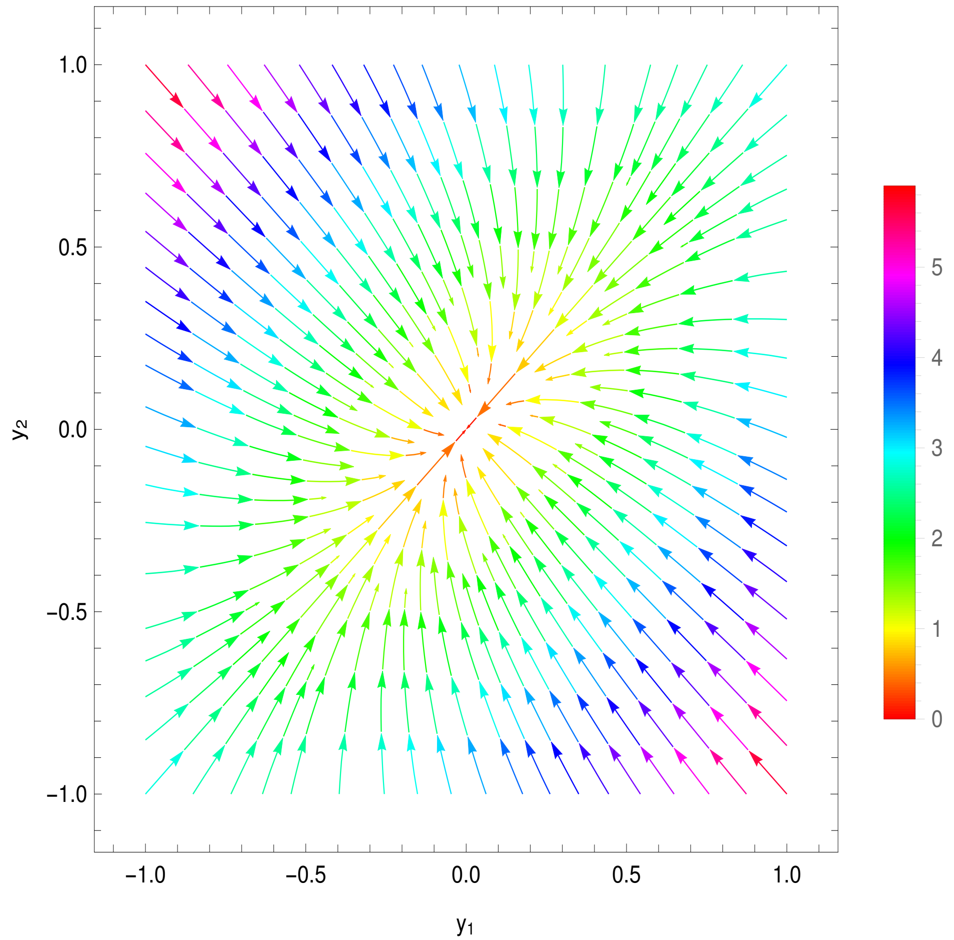

Figure 1. Phase portrait of an improper node

Example 2 (Proper node). $y' = Ay, A = I$.

```
In[22]:= PhasePortrait = StreamPlot[IdentityMatrix[2].{y1,y2},{y1,-1,1},{y2,-1,1},
                            StreamColorFunction->Hue,
             FrameLabel->{Subscript["y",1],Subscript["y",2]},PlotLegends->Automatic];
         Export["/home/student/courses/MATH528/L10Fig02.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L10Fig02.pdf

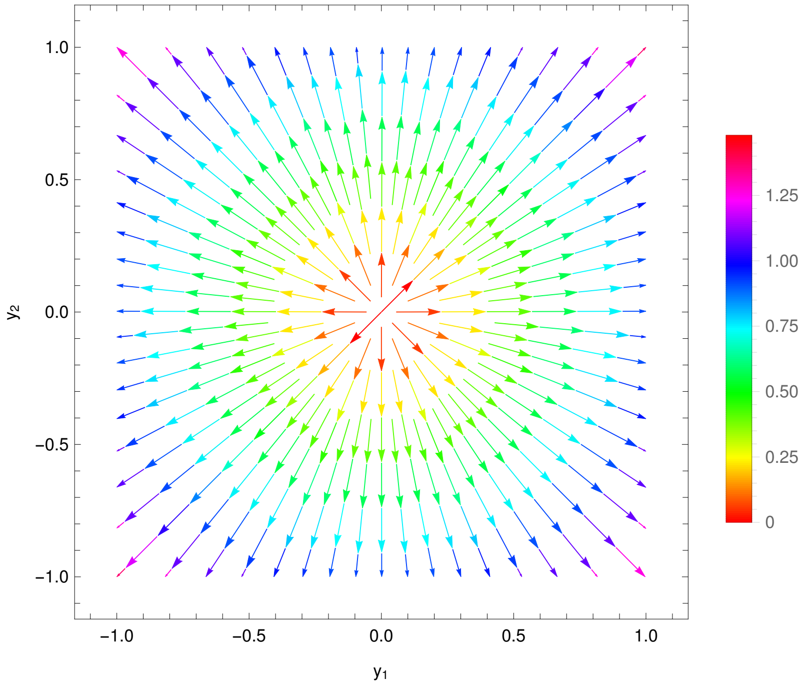

Figure 2. Phase portrait of a proper node

Example 3 (Saddle node). $y' = Ay, A = \mathrm{diag}(1, -1)$.

```
In[23]:= PhasePortrait = StreamPlot[DiagonalMatrix[{1,-1}].{y1,y2},{y1,-1,1},{y2,-1,1},
                             StreamColorFunction->Hue,
                   FrameLabel->{Subscript["y",1],Subscript["y",2]},PlotLegends->Automatic];
         Export["/home/student/courses/MATH528/L10Fig03.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L10Fig03.pdf

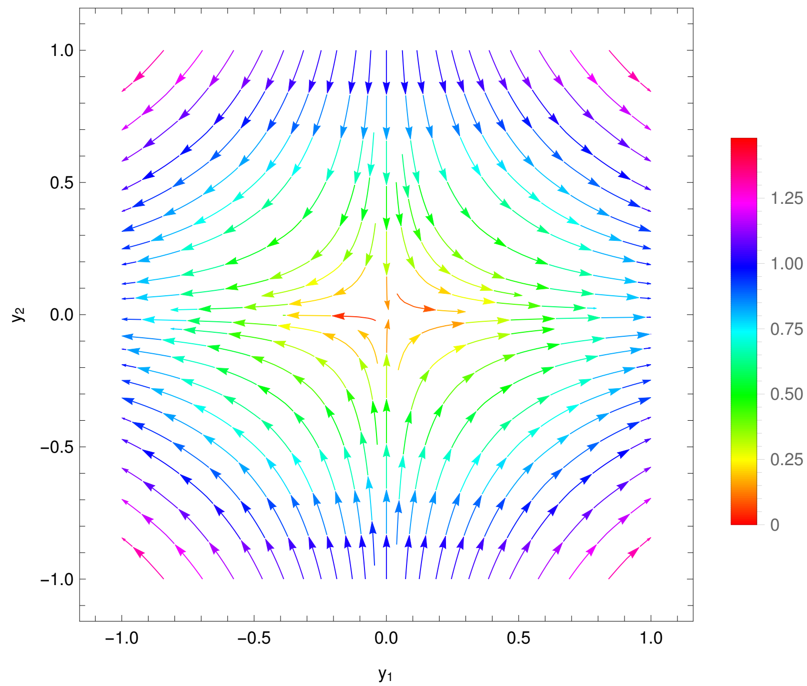

Figure 3. Phase portrait of a saddle node

Example 4 (Center node). $y' = Ay, A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$.

```
In[25]:= PhasePortrait = StreamPlot[{{0,1},{-4,0}}.{y1,y2},{y1,-1,1},{y2,-1,1},
                    StreamColorFunction->Hue,
             FrameLabel->{Subscript["y",1],Subscript["y",2]},PlotLegends->Automatic];
         Export["/home/student/courses/MATH528/L10Fig04.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L10Fig04.pdf

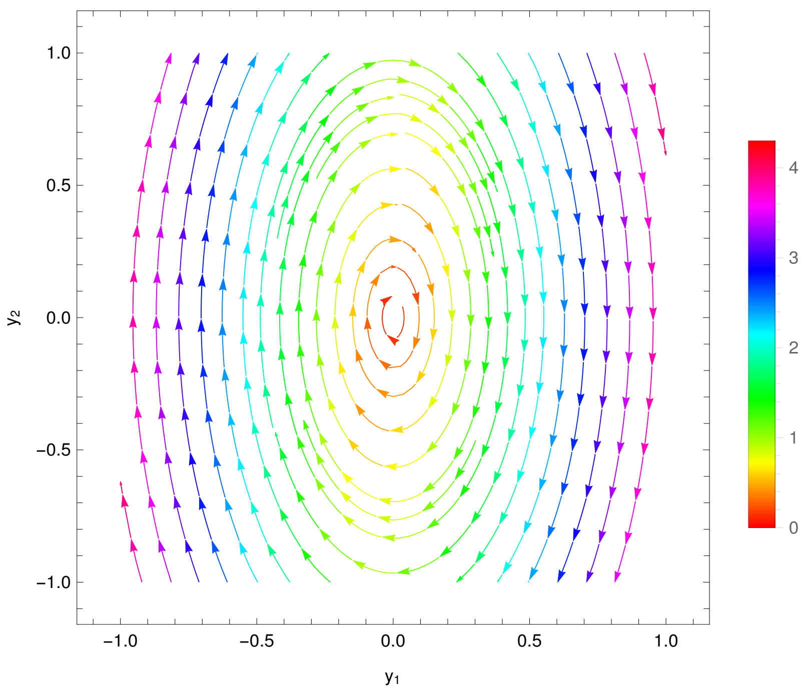

Figure 4. Phase portrait of a center node

Example 5 (Spiral node). $y' = Ay, A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$.

```
In[26]:= PhasePortrait = StreamPlot[{{-1,1},{-1,-1}}.{y1,y2},{y1,-1,1},{y2,-1,1},
                            StreamColorFunction->Hue,
              FrameLabel->{Subscript["y",1],Subscript["y",2]},PlotLegends->Automatic];
         Export["/home/student/courses/MATH528/L10Fig05.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L10Fig05.pdf

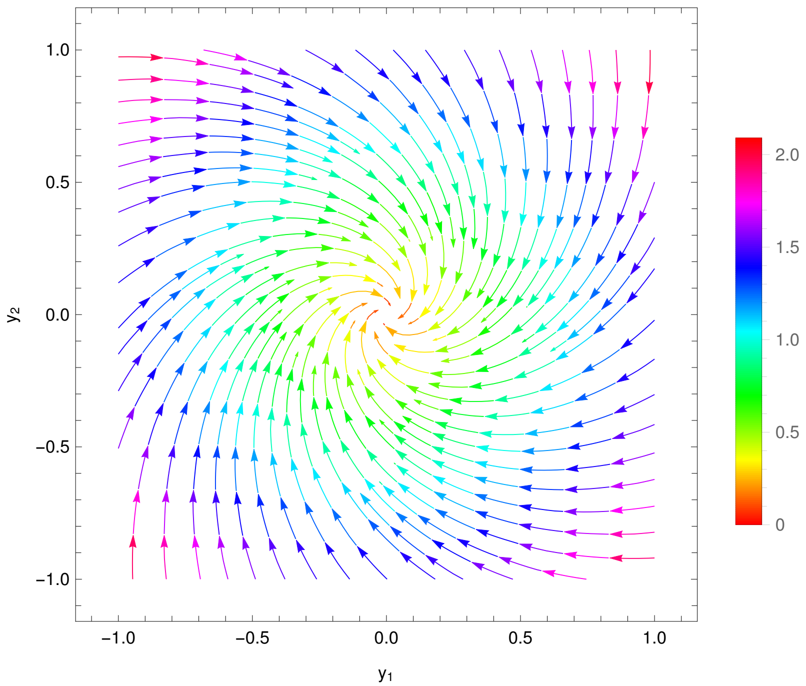

Figure 5. Phase portrait of a spiral node

Example 6 (Degenerate node). $y' = Ay, A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$.

```
In[27]:= PhasePortrait = StreamPlot[{{4,1},{-1,2}}.{y1,y2},{y1,-1,1},{y2,-1,1},
                            StreamColorFunction->Hue,
             FrameLabel->{Subscript["y",1],Subscript["y",2]},PlotLegends->Automatic];
         Export["/home/student/courses/MATH528/L10Fig06.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L10Fig06.pdf

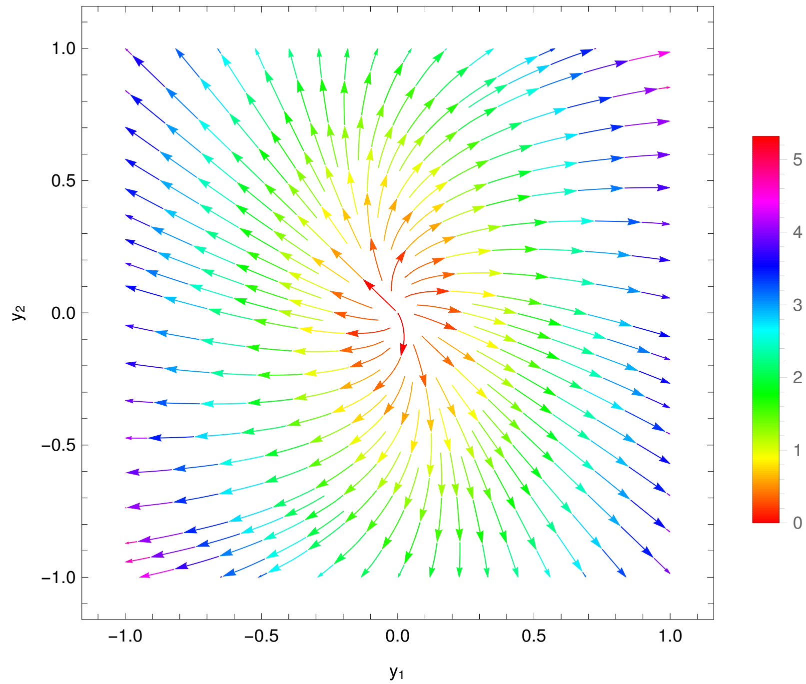

Figure 6. Phase portrait of a center node

Definition. *Roots of* $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{0}$ *are called critical points of the ODE system* $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$.

Definition. *A critical point* $\boldsymbol{y}^*$ *is stable if all trajectories near* $\boldsymbol{y}^*$ *remain close to* $\boldsymbol{y}^*$ *as time increases.*

Definition. *The critical point* $\boldsymbol{y}^*$ *of ODE system* $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$ *is stable if* $\forall \varepsilon > 0$, $\exists \delta_\varepsilon$ *such that if* $\exists t_0$ *such that* $\|\boldsymbol{y}(t_0) - \boldsymbol{y}^*\| < \delta_\varepsilon$, *with* $\boldsymbol{y}(t)$ *a solution of the ODE system, then* $\forall t \geqslant t_0$, $\|\boldsymbol{y}(t) - \boldsymbol{y}^*\| < \varepsilon$. *Otherwise, the critical points is unstable. If, in addition,* $\lim_{t\to\infty} \boldsymbol{y}(t) = \boldsymbol{y}^*$ *for all solutions, the critical point is stable and attractive.*

Algorithm ODE system qualitative analysis

1. For ODE $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$, find roots $\boldsymbol{y}^*$, $\boldsymbol{f}(\boldsymbol{y}^*) = \boldsymbol{0}$.
2. Taylor series expand $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{y}^*) + \boldsymbol{A}(\boldsymbol{y} - \boldsymbol{y}^*) + \mathcal{O}(\|\boldsymbol{y} - \boldsymbol{y}^*\|^2)$
3. Determine type of critical point from eigenvalues of the Jacobian

$$\boldsymbol{A} = \boldsymbol{J}(\boldsymbol{f}(\boldsymbol{y}^*)) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}^*).$$