Algorithm ODE system qualitative analysis

1. For ODE $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$, find roots $\boldsymbol{y}^*$, $\boldsymbol{f}(\boldsymbol{y}^*) = \boldsymbol{0}$.
2. Taylor series expand $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{y}^*) + \boldsymbol{A}\,(\boldsymbol{y} - \boldsymbol{y}^*) + \mathcal{O}(\|\boldsymbol{y} - \boldsymbol{y}^*\|^2)$
3. Determine type of critical point from eigenvalues of the Jacobian

$$\boldsymbol{A} = \boldsymbol{J}(\boldsymbol{f}(\boldsymbol{y}^*)) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}^*).$$

Free undampened pendulum

$$\theta'' + k\sin\theta = 0, (k = g/l)$$

$$\boldsymbol{y}' = \begin{pmatrix} \theta' \\ \omega' \end{pmatrix} = \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} \omega \\ -k\sin\theta \end{pmatrix}$$

Critical points are $\boldsymbol{y}_j^* = \begin{pmatrix} j\pi \\ 0 \end{pmatrix}$. Near a critical point the linearization of the system gives

$$\boldsymbol{y}' = \boldsymbol{A}\,(\boldsymbol{y} - \boldsymbol{y}_j^*), \boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}_j^*) = \begin{pmatrix} 0 & 1 \\ -k\cos(j\pi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{j+1}k & 0 \end{pmatrix}$$

Eigenvalues of $\boldsymbol{A}$:

- Even j: $\lambda_{1,2} = \pm i\sqrt{k}$
- Odd j: $\lambda_{1,2} = \pm\sqrt{k}$

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In[30]:= PhasePortrait = StreamPlot[{omega,-Sin[theta]},{theta,-4Pi,4Pi},{omega,-2Pi,2Pi},
         StreamColorFunction->Hue,FrameLabel->{theta,omega},PlotLegends->Automatic,AspectRatio-
         >Automatic];
         Export["/home/student/courses/MATH528/L11Fig01.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L11Fig01.pdf

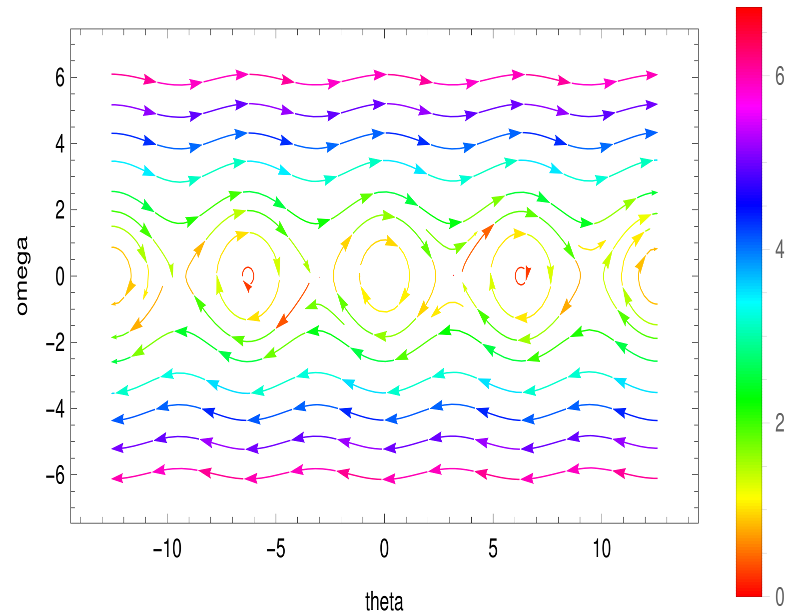

Figure 1. Phase portrait for the free pendulum

Free dampened pendulum

$$\theta'' + c\theta' + k\sin\theta = 0,\ (k = g/l),\ \boldsymbol{y}' = \begin{pmatrix} \theta' \\ \omega' \end{pmatrix} = \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} \omega \\ -k\sin\theta - c\theta \end{pmatrix}$$

Critical points are $\boldsymbol{y}_j^* = \begin{pmatrix} \theta_j \\ 0 \end{pmatrix}$, with $\sin\theta_j = -\frac{c}{k}\theta_j$. Near a critical point the linearization of the system gives

$$\boldsymbol{y}' = \boldsymbol{A}\,(\boldsymbol{y} - \boldsymbol{y}_j^*),\ \boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}_j^*) = \begin{pmatrix} 0 & 1 \\ -k\cos(\theta_j) - c & 0 \end{pmatrix}$$

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In[32]:= PhasePortrait = StreamPlot[{omega,-Sin[theta]-.1 theta},{theta,-4Pi,4Pi},{omega,-2Pi,2Pi},
         StreamColorFunction->Hue,FrameLabel->{theta,omega},PlotLegends->Automatic,AspectRatio-
         >Automatic]; Export["/home/student/courses/MATH528/L11Fig02.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L11Fig02.pdf

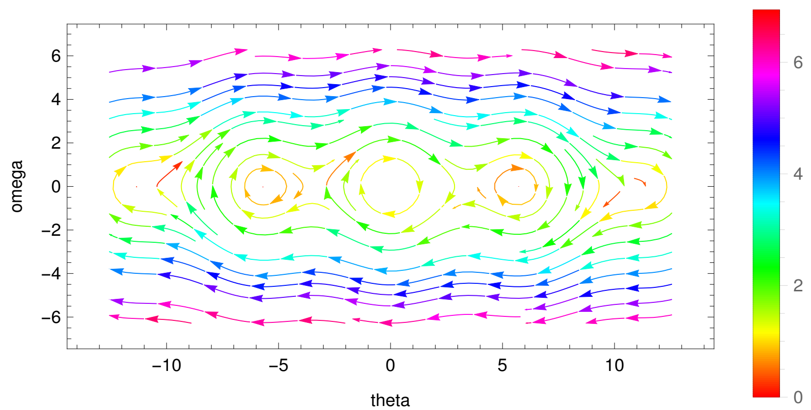

Figure 2. Phase portrait for the dampened pendulum

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} a y_1 - b y_1 y_2 \\ k y_1 y_2 - l y_2 \end{pmatrix}; \boldsymbol{J} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} a - b y_2 & -b y_1 \\ k y_2 & k y_1 - l \end{pmatrix}$$

```
In[38]:= PhasePortrait = StreamPlot[{y1 -  y1 y2, y1 y2 - y2},{y1,0,5},{y2,0,5},StreamColorFunction->Hue,
         FrameLabel->{"y1","y2"},PlotLegends->Automatic,AspectRatio->Automatic];
         Export["/home/student/courses/MATH528/L11Fig03.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L11Fig03.pdf

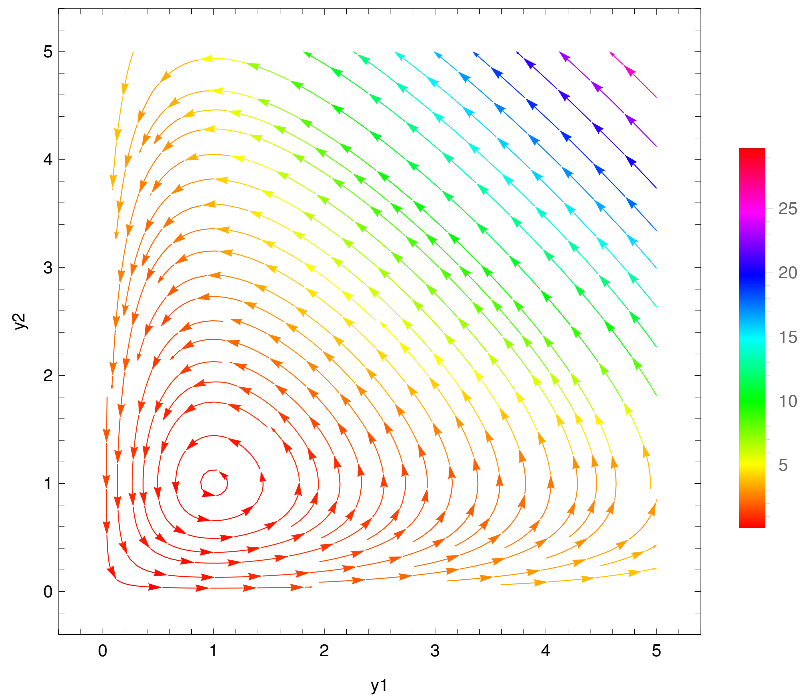

Figure 3.

$$z'' - \mu(1-z^2)z' + z = 0,\ \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z \\ z' \end{pmatrix},\ \boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} y_2 \\ \mu(1-y_1^2)y_2 - y_1 \end{pmatrix}$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} 0 & 1 \\ -2\mu y_1 y_2 - 1 & \mu(1-y_1^2) \end{pmatrix}$$

```
In[39]:= PhasePortrait = StreamPlot[{y2, (1-y1^2) y2 - y1},{y1,-5,5},{y2,-5,5},StreamColorFunction->Hue,
         FrameLabel->{"y1","y2"},PlotLegends->Automatic,AspectRatio->Automatic];
         Export["/home/student/courses/MATH528/L11Fig04.pdf",PhasePortrait]
```

/home/student/courses/MATH528/L11Fig04.pdf

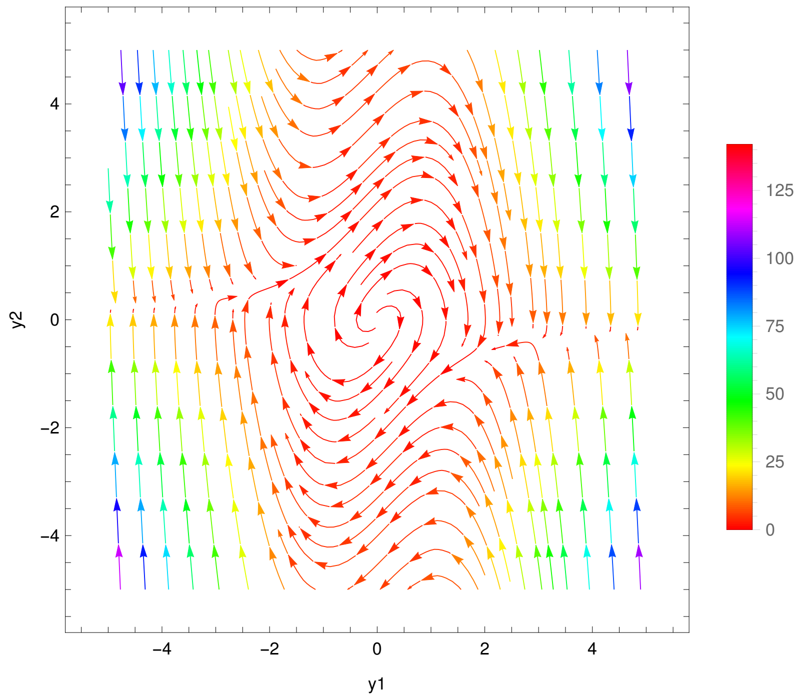

Figure 4.