Remark. The real numbers are a complete, ordered field $(\mathbb{R},+,\times)$

Remark. Power series are simply an infinite sequence of the operations defined in $\mathbb{R}$

$$S_n(x)=\sum_{j=0}^{n} a_j x^j, n\in\mathbb{N}$$

Remark. Power series can also be interpreted as a sequence of scalar products

$$S_n(x)=\boldsymbol{a}^T\boldsymbol{b}(x), \boldsymbol{a}^T=\begin{pmatrix} a_0 & a_1 & \ldots & a_n \end{pmatrix}, \boldsymbol{b}(x)=\begin{pmatrix} 1 & x & \ldots & x^n \end{pmatrix}^T$$

The power series method to solve ODEs onsist of:

1. Introducing a representation $y(x)=\sum_{j=0}^{\infty} a_j x^j$
2. Replacing the representation into the ODE of interest and identifying coefficients of powers of x

Example. $y'=y$, $y(0)=y_0$, Try $y=a_0+a_1x+\cdots+a_nx^n+\cdots+$

$$\begin{array}{llll} y'=a_1+2a_2x+\cdots+(n+1)a_{n+1}x^n & = & & y=a_0+a_1x+\cdots+a_nx^n+\cdots+=y\Rightarrow \\ a_1=a_0, & & a_2=\frac{1}{2}a_1,\ldots & a_{n+1}=\frac{1}{n+1}a_n\Rightarrow \\ y(x)= & \left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\ldots\right)a_0 & & =e^x y(0), a_0=y(0). \end{array}$$

Example. $y''=-y$, Try $y=a_0+a_1x+\cdots+a_nx^n+\cdots+$

$$\begin{aligned}
y''=2\cdot 1\cdot a_2+3\cdot 2\cdot a_3\,x+\cdots+(n+2)\cdot(n+1)\cdot a_{n+2}x^n &= -(a_0+a_1x+\cdots+a_nx^n+\cdots)=y\Rightarrow\\
a_2=-\frac{1}{2\cdot 1}a_0,\, a_3=-\frac{1}{3\cdot 2}a_1,\\
y(x)=a_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots-\right) &+ a_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots-\right)\\
y(x) &= a_0\cos(x)+a_1\sin(x)
\end{aligned}$$

Remark. The equation $Y''+Y=0$ results from separation of variables applied to $\nabla^2u=0$, $u(x,y,z)=X(x)Y(y)Z(z)$ expressed in Cartesian coordinates (x,y,z).

Recall:

$$\begin{aligned}
\nabla &= \partial_x\,\boldsymbol{e}_x+\partial_y\,\boldsymbol{e}_y+\partial_z\,\boldsymbol{e}_z\\
\nabla u=\operatorname{grad}u &= \partial_xu\,\boldsymbol{e}_x+\partial_yu\,\boldsymbol{e}_y+\partial_zu\,\boldsymbol{e}_z=\boldsymbol{v}=v_x\boldsymbol{e}_x+v_y\boldsymbol{e}_y+v_z\boldsymbol{e}_z\\
\nabla\cdot\boldsymbol{v}=\operatorname{div}\boldsymbol{v} &= \partial_xv_x+\partial_yv_y+\partial_zv_z\\
\nabla^2u &= \partial_x^2u+\partial_y^2u+\partial_z^2u
\end{aligned}$$

$$\operatorname{div}\boldsymbol{v}=\lim_{|\Omega|\to 0}\frac{\oint_{\partial\Omega}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\sigma}}{|\Omega|},\ \partial\Omega=\text{boundary of }\Omega.$$

The radius of convergence can be determined from series coefficients:

$$R^{-1} = \lim_{n\to\infty} |a_n|^{1/n} \quad R^{-1} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

if limits exist and $R \neq 0$.

Example. $e^x = 1 + x + \cdots + \frac{x^n}{n!}$, $R^{-1} = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow R = \infty$, converges $\forall x$

Example. $\frac{1}{1-x} = 1 + x + x^2 + \cdots +, R^{-1} = 1 \Rightarrow R = 1$, converges for $|x| < 1$

In general a series solution with $R > 0$ exists for ODEs of form $y'' + p(x)y' + q(x)y = 0$ if p, q have power series representations.

Remark. $\sin(x), \cos(x)$ obtained from separation of variables applied to $\nabla^2 u = 0$ in Cartesian

Similarly useful functions are obtained through series solutions to the ODE obtained by separation of variables of $\nabla^2 u = 0$ in other coordinate systems: spherical, cylindrical, ...

$$x = r\cos\theta\sin\varphi,\, y = r\sin\theta\sin\varphi,\, z = r\cos\varphi$$

$$L_r = 1,\, L_\varphi = r,\, L_\theta = r\sin\varphi\ (\text{Lame coefficients})$$

$$\operatorname{grad} u = \frac{\partial u}{L_r\,\partial r}\boldsymbol{e}_r + \frac{\partial u}{L_\varphi\,\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial u}{L_\theta\,\partial\theta}\boldsymbol{e}_\theta$$

$$\operatorname{div}\boldsymbol{v} = \frac{1}{L_r L_\varphi L_\theta}\left[\frac{\partial}{\partial r}(v_r L_\varphi L_\theta) + \frac{\partial}{\partial\varphi}(L_r v_\varphi L_\theta) + \frac{\partial}{\partial\theta}(L_r L_\varphi v_\theta)\right]$$