- Recall:
  - $\lambda, e^{\lambda x}$ are eigenvalues, eigenfunctions of the differentiation operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$

$$D e^{\lambda t} = \lambda e^{\lambda t}$$

  - Constant coefficient ODEs

$$D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = 0$$

    have a general solution

$$y(t) = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t},$$

    a linear combination of eigenfunctions of D with $r_1, \ldots, r_n$ solutions of the algebraic, characteristic eq.

$$r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

- Idea: seek a more general linear combination of eigenfunctions e^{-st} with a continuous range of eigenvalues
  - denote the coefficients of the linear combination as $f(t)$
  - choose $s \geqslant 0$, and e^{-st} to avoid exponentially increasing functions
  - replace summation of $e^{r_i t}$ for a finite linear combination with integration over s values

Definition. *The Laplace transform of $f: [0, \infty) \rightarrow \mathbb{R}$ is*

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}t, F = \mathcal{L}(f),$$

and $f(t) = \mathcal{L}^{-1}(F)(t)$, $f = \mathcal{L}^{-1}(F)$ is the inverse Laplace transform.

Definition. *The integral transform of* $f\colon\mathcal{D}\to\mathbb{R}$ *with kernel* $k(s,t)$ *is*

$$F(s)=\int_{\mathcal{D}} f(t)\,k(s,t)\,\mathrm{d}t.$$

Example 1. $f(t)=1$ for $t\geqslant 0$,

$$\mathcal{L}(f)(s)=\int_0^\infty e^{-st}\mathrm{d}t=\lim_{T\to\infty}\int_0^T e^{-st}\mathrm{d}t=\lim_{T\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^T=\frac{1}{s}$$

```
In[2]:= LaplaceTransform[1,t,s]
```

$$\frac{1}{s}$$

Example 2. $f(t)=e^{at}$ for $t\geqslant 0$,

$$\mathcal{L}(f)(s)=\int_0^\infty e^{-(s-a)t}\mathrm{d}t=\lim_{T\to\infty}\int_0^T e^{-(s-a)t}\mathrm{d}t=\lim_{T\to\infty}\left[-\frac{e^{-st}}{s-a}\right]_0^T=\frac{1}{s-a}$$

```
In[3]:= LaplaceTransform[Exp[a t],t,s]
```

$$\frac{1}{s-a}$$

```
In[4]:=
```

Theorem. $\mathcal{L}(af+bg)=a\mathcal{L}(f)+b\mathcal{L}(g)$

Example 3. $f(t)=\cosh(at)=\frac{1}{2}(e^{at}+e^{-at})$, $g(t)=\sinh(at)=\frac{1}{2}(e^{at}-e^{-at})$,

$$\mathcal{L}(f)=\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{s}{s^2-a^2}, \mathcal{L}(g)=\frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{a}{s^2-a^2}$$

```
In[4]:= LaplaceTransform[{Cosh[a t],Sinh[a t]},t,s]
```

$$\left\{\frac{s}{s^2-a^2},\frac{a}{s^2-a^2}\right\}$$

Example 4. $f(t)=\cos(\omega t)=\frac{1}{2}(e^{i\omega t}+e^{-i\omega t})$, $g(t)=\sin(at)=\frac{1}{2i}(e^{i\omega t}-e^{-i\omega t})$,

$$\mathcal{L}(f)=\frac{1}{2}\left(\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right)=\frac{s}{s^2+\omega^2}, \mathcal{L}(g)=\frac{1}{2i}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{\omega}{s^2+\omega^2}$$

```
In[5]:= LaplaceTransform[{Cos[omega t],Sin[omega t]},t,s]
```

$$\left\{\frac{s}{\omega^2+s^2},\frac{\omega}{\omega^2+s^2}\right\}$$

```
In[6]:=
```

Theorem. $F=\mathcal{L}(f)\Rightarrow F(s-a)=\mathcal{L}[e^{at}f(t)]$, $e^{at}f(t)=\mathcal{L}^{-1}[F(s-a)]$

Example 5. Inverse Laplace transform of

$$\mathcal{L}(f)=\frac{3s-137}{s^2+2s+401}=\frac{3(s+1)-140}{(s+1)^2+400}=3\frac{s+1}{(s+1)^2+20^2}-7\frac{20}{(s+1)^2+20^2}$$

$$f=3\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+20^2}\right]-\mathcal{L}^{-1}\left[\frac{20}{(s+1)^2+20^2}\right]=e^{-t}[3\cos(20t)-7\sin(20t)].$$

```
In[8]:= Simplify[ComplexExpand[InverseLaplaceTransform[(3s-137)/(s^2+2s+401),s,t]]]
```

$e^{-t}(3\cos(20t)-7\sin(20t))$

Theorem. $f\colon[0,\infty)\to\mathbb{R}$, *piecewise continuous and* $\exists M,k$ *such that* $|f(t)|\leqslant Me^{kt}$, *then* $\mathcal{L}(f)$ *exists.*