Definition. *The* *Laplace transform* *of* $f\colon [0,\infty)\to\mathbb{R}$ *is*

$$F(s)=\mathcal{L}(f)(s)=\int_0^\infty f(t)\,e^{-st}\,\mathrm{d}t,\, F=\mathcal{L}(f),$$

and $f(t)=\mathcal{L}^{-1}(F)(t)$, $f=\mathcal{L}^{-1}(F)$ *is the* *inverse Laplace* *transform.*

Theorem. $\mathcal{L}(f')=s\,\mathcal{L}(f)-f(0)$, $\mathcal{L}(f'')=s^2\,\mathcal{L}(f)-s\,f(0)-f'(0)$,

$$\mathcal{L}(f^{(n)})=s^n\,\mathcal{L}(f)-\sum_{j=0}^{n-1} s^j\,f^{(n-1-j)}(0)$$

Theorem.

$$\mathcal{L}\left(\int_0^t f(\tau)\,\mathrm{d}\tau\right)=\frac{1}{s}F(s),\ \int_0^t f(\tau)\,\mathrm{d}\tau=\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$$

Example.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right)=\frac{\sin\omega t}{\omega}\Rightarrow\mathcal{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right)=\int_0^t\frac{\sin\omega\tau}{\omega}\,\mathrm{d}\tau=\frac{1-\cos\omega t}{\omega^2}$$

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In[1]:= InverseLaplaceTransform[1/s/(s^2+omega^2),s,t]
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$$\frac{1-\cos(\omega\,t)}{\omega^2}$$

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In[2]:=
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$$y'' + ay' + by = r(t),\, y(0) = K_0,\, y'(0) = K_1$$

1. Apply Laplace transform: $s^2Y - sy(0) - y'(0) + a[sY - y(0)] + bY = R(s) \Rightarrow$

$$(s^2 + as + b)Y = (s + a)K_0 + K_1 + R(s)$$

2. Define the *transfer function* $Q(s)$

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{\left(s + \frac{1}{2}a\right)^2 + b - \frac{1}{4}a^2}$$

   and find solution

$$Y = [(s + a)K_0 + K_1]Q(s) + R(s)Q(s)$$

   If $K_0 = K_1 = 0$ then

$$Q(s) = \frac{Y(s)}{R(s)} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}\,.$$

3. Invert Y to find $y = \mathcal{L}^{-1}(Y)$

Advantages

- Directly finds solution to nonhomogeneous problems
- Accounts for initial values
- Efficient representation of rhs inputs $r(t)$