Definition. *The Heaviside (step) function* $u(t-a)$ *is defined as*

$$u(t-a)=\begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}.\ (a\geqslant 0),\ \mathcal{L}[u(t-a)]=\frac{e^{-as}}{s}$$

Definition. *The impulse of force* $f(t)$ *over time interval* $[a,a+k]$ *is the integral*

$$I=\int_a^{a+k} f(t)\,\mathrm{d}t.$$

Consider k small such that $f(t)$ is approximately constant and described by

$$f_k(t-a)=\begin{cases} 1/k & a\leqslant t\leqslant a+k \\ 0 & \text{otherwise} \end{cases},\ I_k=\int_a^{a+k} f_k(t)\,\mathrm{d}t=1.$$

Note that $f_k(t-a)=\frac{1}{k}[u(t-a)-u(t-a-k)]$ such that

$$F_k(s)=\mathcal{L}(f_k)(s)=\frac{1}{k}\left[\frac{e^{-as}}{s}-\frac{e^{-(a+k)s}}{s}\right],\ \lim_{k\to 0} F_k(s)=e^{-as}=\mathcal{L}[\delta(t-a)]$$

Definition. *The (generalized) function* $\lim_{k\to 0} f_k(t)=\delta(t-a)=\mathcal{L}^{-1}[e^{-as}]$ *is the Dirac delta function.*

Properties:

$$\int_0^{\infty} \delta(t-a)\,\mathrm{d}t=1 \quad \int_0^{\infty} g(t)\delta(t-a)\,\mathrm{d}t=g(a)$$