Recall that the Laplace transform is linear: $\alpha, \beta \in \mathbb{R}$, $f, g: [0, \infty) \to \mathbb{R}$

$$H = \mathcal{L}(h) = \mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g) = \alpha F + \beta G.$$

- The linearity property allows solution of constant-coefficient ODEs, e.g. $y'' + a y' + b y = r(t)$
- What about variable-coefficient ODEs, e.g., $y'' + p(t)\, y' + q(t)\, y = r$?
- This question leads to consideration of $\mathcal{L}(fg)$, and $\mathcal{L}^{-1}(FG)$.

**Theorem.** *Consider $f, g: [0, \infty) \to \mathbb{R}$, with Laplace transforms $F(s) = \mathcal{L}(f)(s), G(s) = \mathcal{L}(g)(s)$. The product $H(s) = F(s) G(s) = \mathcal{L}(h)$ is the Laplace transform of $h(t) = (f * g)(t)$, where the convolution product $f * g$ is*

$$h(t) = (f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, \mathrm{d}\tau.$$

***Proof.*** $\to$ *Write Laplace transforms as* $F(s) = \int_0^\infty e^{-s\tau} f(\tau) \mathrm{d}\tau$, $G(s) = \int_0^\infty e^{-sp} g(p) \mathrm{d}p$

$\to$ *Change integration variable* $p = t - \tau$, $G(s) = \int_\tau^\infty e^{-s(t-\tau)} g(t - \tau) \mathrm{d}t = e^{s\tau} \int_\tau^\infty e^{-st} g(t - \tau) \mathrm{d}t$

$\to$ $(t, \tau)$ *are independent hence*

$$F(s) G(s) = F(s) \int_\tau^\infty e^{-s(t-\tau)} g(t - \tau) \mathrm{d}t = \int_\tau^\infty F(s) e^{-s(t-\tau)} g(t - \tau) \mathrm{d}t =$$

$$= \int_\tau^\infty \left( \int_0^\infty e^{-s\tau} f(\tau) \mathrm{d}\tau \right) e^{-s(t-\tau)} g(t - \tau) \mathrm{d}t = \int_0^\infty e^{-st} \left[ \int_0^t f(\tau) g(t - \tau)\, \mathrm{d}\tau \right] \mathrm{d}t$$

□

**Example 1.** $H(s)=\frac{1}{s(s-a)}=F(s)G(s),\ F(s)=\frac{1}{s},\ G(s)=\frac{1}{s-a},h(t)=\mathcal{L}^{-1}[H(s)]=(f*g)(t)$

$$f(t)=\mathcal{L}^{-1}\left(\frac{1}{s}\right)=1,\, g(t)=\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=e^{at},\, h(t)=\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau=\int_0^t e^{a(t-\tau)}\mathrm{d}\tau=\frac{1}{a}(e^{at}-1)$$

```
In[6]:= InverseLaplaceTransform[1/s/(s-a),s,t]
```

$$\frac{e^{a\,t}-1}{a}$$

```
In[7]:=
```

**Example 2.** $H(s)=\frac{1}{(s^2+\omega^2)^2}=F(s)F(s),\ F(s)=\frac{1}{s^2+\omega^2},h(t)=\mathcal{L}^{-1}[H(s)]=(f*f)(t)$

$$f(t)=\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right)=\frac{\sin\omega t}{\omega},\, h(t)=\int_0^t f(\tau)f(t-\tau)\mathrm{d}\tau=\frac{1}{\omega^2}\int_0^t \sin[\omega\tau]\sin[\omega(t-\tau)]\mathrm{d}\tau=$$

$$\frac{1}{2\omega^2}\left[\int_0^t \cos(2\omega\tau-\omega t)\mathrm{d}\tau-\int_0^t \cos(\omega t)\mathrm{d}\tau\right]=\frac{1}{2\omega^2}\left[\left[\frac{\sin(\omega(2\tau-t))}{2\omega}\right]_{\tau=0}^{\tau=t}-t\cos(\omega t)\right]=$$

$$\frac{1}{2\omega^2}\left[\frac{\sin(\omega t)}{\omega}-t\cos(\omega t)\right]$$

```
In[8]:= InverseLaplaceTransform[1/(s^2+omega^2)^2,s,t]
```

$$\frac{\sin(\omega\,t)-\omega\,t\cos(\omega\,t)}{2\,\omega^3}$$

**Example 3.** Solve by Laplace transform

$$y(t)-\int_0^t y(\tau)(t-\tau)\,\mathrm{d}\tau = y(t)-\int_0^t y(\tau)g(t-\tau)\mathrm{d}\tau = 2-\frac{t^2}{2},\ g(t)=t \tag{1}$$

Take $\mathcal{L}\,(1)$:

$$Y(s)-Y(s)\frac{1}{s^2}=\frac{s^2-1}{s^2}Y(s)=\frac{2}{s}-\frac{1}{2}\frac{2}{s^3}=\frac{1}{s}\left(2-\frac{1}{s^2}\right)=\frac{2s^2-1}{s^3}\Rightarrow Y(s)=\frac{2s^2-1}{s(s^2-1)}$$

Separte into partial fractions

$$Y(s)=\frac{1}{s}+\frac{1}{2\,(s+1)}+\frac{1}{2\,(s-1)}\Rightarrow y(t)=1+\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}=1+\cosh(t)$$

Let

$$F(s)=\frac{2s^2-1}{s};G(s)=\frac{1}{(s-1)(s+1)}\Rightarrow f(t)=, g(t)=$$

```
In[28]:= InverseLaplaceTransform[1,s,t]
```

$\delta(t)$

```
In[23]:= Expand[InverseLaplaceTransform[Apart[Y /. Solve[Y (1-1/s^2) == 2/s - 1/s^3, Y][[1,1]]],s,t]]
```

$$\frac{e^{-t}}{2}+\frac{e^t}{2}+1$$

```
In[24]:= Y /. Solve[Y (1-1/s^2) == 2/s - 1/s^3, Y][[1,1]]
```

$$\frac{2\,s^2-1}{s\,(s^2-1)}$$

In[27]:= LaplaceTransform[UnitStep[t-a],t,s]

$$\frac{e^{-a\,s}\,\theta(a)+\theta(-a)}{s}$$

In[30]:= LaplaceTransform[Exp[-(t-a)^2],t,s]

$$\frac{1}{2}\sqrt{\pi}\,e^{\frac{1}{4}s\,(s-4a)}\left(\operatorname{erf}\left(a-\frac{s}{2}\right)+1\right)$$

In[31]:=

**Example 4.**

$$H(s) = \frac{1}{(s-a)^2} = F(s)F(s),\, F(s) = \frac{1}{s-a} \Rightarrow f(t) = e^{at}$$

$$h(t) = (f * f)(t) = \int_0^t f(\tau) f(t-\tau)\,\mathrm{d}\tau = \int_0^t e^{a\tau} e^{a(t-\tau)}\,\mathrm{d}\tau = \int_0^t e^{at}\,\mathrm{d}\tau = t\,e^{at}$$

```
In[32]:= InverseLaplaceTransform[1/(s-a)^2,s,t]
```

$t\,e^{a\,t}$

```
In[33]:=
```