- The utility of the orthonormal trigonometric basis set $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$ motivates the search for a general procedure to generate families of orthonormal functions
- Searching for periodic-in-time solutions of the vibrating string equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0,$$

in the form $y(t, x) = e^{i\omega t} z(x)$ leads to the ODE

$$z'' + k^2 z = 0$$

with $k^2 = \omega^2 / c^2$. Imposing boundary conditions such as $z(0) = 0$, $z(L) = 0$, (fixed ends) allows non-trivial solutions only for specific choices of $k = \pi j / L$, $j \in \mathbb{Z}$, known as eigenvalues. Consideration of variable string thickness, possible dampening, variable storage properties leads, in conjunction with the desire to obtain solutions that form an orthonormal set leads to the definition of a Sturm-Liouville problem.

Definition. *The second-order, homogeneous, linear problem for* $y: [a, b] \to \mathbb{R}$ *defined by*

$$\begin{cases} (p(x) y')' + (q(x) + \lambda r(x))\, y = 0 \\ k_1 y(a) + k_2 y'(a) = 0 \\ l_1 y(b) + l_2 y'(b) = 0 \end{cases}, \tag{1}$$

with $p, q, r: [a, b] \to \mathbb{R}$, *and* $\lambda, k_1, k_2, l_1, l_2 \in \mathbb{R}$ *is known as a Sturm-Liouville problem.*

Clearly $y(x) = 0$ is a trivial solution. The interest is in possible non-trivial solutions.