Solution of the Sturm-Liouville problem for $y\colon [a,b] \to \mathbb{R}$ defined by

$$\begin{cases} (p(x)y')' + (q(x) + \lambda r(x))\, y = 0 \\ k_1 y(a) + k_2 y'(a) = 0 \\ l_1 y(b) + l_2 y'(b) = 0 \end{cases}, \tag{1}$$

with $p, q, r\colon [a,b] \to \mathbb{R}$, and $\lambda, k_1, k_2, l_1, l_2 \in \mathbb{R}$, gives a family of orthogonal functions $\{y_0, y_1, \ldots\}$

Definition. *Given a family of functions $\mathcal{B} = \{y_0, y_1, \ldots\}$ (a basis set), defined on $[a,b]$, orthogonal w.r.t the scalar product*

$$(f, g) = \int_a^b r(x)\, f(x)\, g(x)\, \mathrm{d}x,$$

and a function $f\colon [a,b] \to \mathbb{R}$, a convergent series of the form

$$f(x) = \sum_{m=0}^{\infty} a_m\, y_m(x),$$

is known as the orthogonal expansion of f on $\mathcal{B}$. The scalar coefficients a_m are known as the Fourier coefficients w.r.t. the basis set $\mathcal{B}$, and are determined as

$$a_m = \frac{(f, y_m)}{(y_m, y_m)} = \frac{1}{\|y_m\|^2} \int_a^b f(x)\, y_m(x)\, \mathrm{d}x,$$

or as

$$a_m = (f, y_m)$$

for an orthonormal basis set that has $\|y_m\| = 1$.