Express the 3D wave equation $u_{tt} - c^2\nabla^2 u = 0$ (subscripts denote differentiation) in spherical coordinates

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi$$

$$\frac{1}{c^2}u_{tt} = \frac{1}{r^2}(r^2u_r)_r + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} + \frac{1}{r^2\sin\phi}(\sin\phi\, u_\phi)_\phi$$

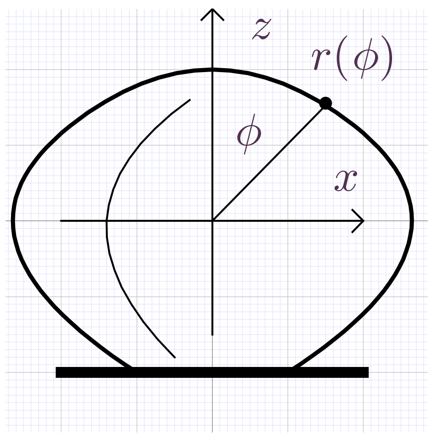

Figure 1.

Seek time periodic solutions, and apply separation of variables $u(t,r,\theta,\phi) = a\,e^{i\omega t}R(r)\Theta(\theta)\Phi(\phi)$ to obtain

$$-\frac{\omega^2}{c^2}R\,\Theta\,\Phi = R''\,\Theta\,\Phi + \frac{2}{r}R'\,\Theta\,\Phi + \frac{1}{r^2\sin^2\phi}R\,\Theta''\,\Phi + \frac{1}{r^2}R\,\Theta\,\Phi'' + \frac{\cos\phi}{r^2\sin\phi}R\,\Theta\,\Phi'$$

$$-\frac{\Theta''}{\Theta} = \frac{r^2\sin^2\phi}{R}\left(R'' + \frac{2}{r}R' + \frac{\omega^2}{c^2}R\right) + \sin^2\phi\left(\frac{\Phi''}{\Phi} + \cot\phi\frac{\Phi'}{\Phi}\right) = m^2$$

$$\frac{r^2}{R}\left(R'' + \frac{2}{r}R' + \frac{\omega^2}{c^2}R\right) + \left(\frac{\Phi''}{\Phi} + \cot\phi\frac{\Phi'}{\Phi} - \frac{m^2}{\sin^2\phi}\right) = 0$$

ODEs along (r,θ) are obtained, and restated in Sturm-Liouville eigenrelation form $\frac{1}{w}[-(py')'+qy]=\lambda y$

$$r^2 R''+2rR'+[r^2k^2-l(l+1)]R=0 \Leftrightarrow \frac{1}{r^2}[-(r^2 R')'+l(l+1)R]=k^2R \Rightarrow w=r^2$$

$$\Theta''+m^2\Theta=0 \Leftrightarrow -(\Theta')'=m^2\Theta \Rightarrow w=1$$

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In[6]:= DSolve[r^2 R''[r] + 2r R'[r] + (r^2 k^2 - l(l+1)) R[r] == 0, R[r], r][[1,1]]
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$R(r)\to c_1\, j_l\,(k\,r)+c_2\, y_l\,(k\,r)$

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In[11]:= DSolve[THETA''[theta]+m^2 THETA[theta]==0, THETA[theta], theta][[1,1]]
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$\Theta(\theta)\to c_2\sin(m\,\theta)+c_1\cos(m\,\theta)$

The ODE along ϕ is restated through the transformation of variables $x=\cos\phi$, $\Phi(\phi)=P(x)=P(\cos\phi)$

$$\Phi'=\frac{\mathrm{d}\Phi}{\mathrm{d}\phi}=\frac{\mathrm{d}P(\cos\phi)}{\mathrm{d}\phi}=\frac{\mathrm{d}P(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\phi}=-\sin\phi P',$$

$$\Phi''=-\cos\phi P'+\sin^2\phi P''=(1-x^2)P''-xP',$$

$$\Phi''+\cot\phi\Phi'-\frac{m^2}{\sin^2\phi}\Phi=(1-x^2)P''-xP'-xP'-\frac{m^2}{1-x^2}P=-l(l+1)P,$$

$$\left[-((1-x^2)P')'+\frac{m^2}{1-x^2}P\right]=l(l+1)P \Rightarrow w=1$$

with solution

$$\Phi(\phi)=P_l^m(\cos\theta)$$