MATH529

SP.22: Homework 3

Instructions

This is a first homework assignment on use of the concepts of linearity and separability to solve linear partial differential equations in two variables.

Exercises

From p.725: 2-6

Model solution: 13.2.1 (p.725)

Statement

A rod of length L with initial temperature $f(x)$ is held at left at temperature zero and at right is insulated. State initial boundary value problem for heat equation.

Solution

The IBVP is:
$u_t = k\, u_{xx}$ for $0 < x < L,\ t > 0$
$u(x, t = 0) = f(x)$ for $0 \leq x \leq L$
$u(x = 0, t) = 0$ for $t > 0$
$u_x(x = L, t) = 0$ for $t > 0$

Note: Pay careful attention to the definition of the domain of validity of the PDE and the initial and boundary value conditions:
(1) Derivatives cannot be defined at endpoints
(2) Initial conditions can be discontinuous w.r.t. boundary conditions

In[1]:=
```
PDE = ∂_t u[x, t] == k ∂_{x,x} u[x, t];
IC = u[x, 0] == f[x];
BClft = u[0, t] == 0;
BCrgt = ((∂_x u[x, t]) /. x → L) == 0;
```

In[5]:=
```
IBVP = {PDE, IC, BClft, BCrgt}
```

Out[5]=
```
{u^(0,1)[x, t] == k u^(2,0)[x, t], u[x, 0] == f[x], u[0, t] == 0, u^(1,0)[L, t] == 0}
```

Problems

From pp.727-8:2-8

Model solution: 13.3.1 (p. 727)

Statement

Solve the IBVP:

$u_t = k\, u_{xx}$ for $0 < x < L,\ t > 0$

$u(x, t = 0) = 100$ for $0 \le x \le L$

$u(x = 0, t) = 0$ for $t > 0$

$u_x(x = L, t) = 0$ for $t > 0$

Solution

This is a heat equation with homogeneous boundary conditions and has a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{\pi n x}{L} \exp\left[-k\left(\frac{\pi n}{L}\right)^2 t\right]$$

with coefficients A_n given by the sine series of the initial condition

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{\pi n x}{L}\, dx = \frac{200}{L} \int_0^L \sin \frac{\pi n x}{L}\, dx = -\frac{200}{\pi n}\left[\cos \frac{\pi n x}{L}\right]_{x=0}^{x=L} \Rightarrow$$

$$A_n = \frac{200}{\pi n}\left(1 - (-1)^n\right)$$

The initial condition is discontinuous w.r.t. the homogeneous Dirichlet condition, hence the sine series coefficients decay slowly as n^{-1} and Gibbs phenomenon is observable in the formulation of the initial condition. However, for $t > 0$ Gibbs oscillations disappear rapidly.

Solution visualization

In[6]:=
```
A[n_, L_] = Assuming[n ∈ Integers, Integrate[Sin[π n x / L], {x, 0, L}]]
```

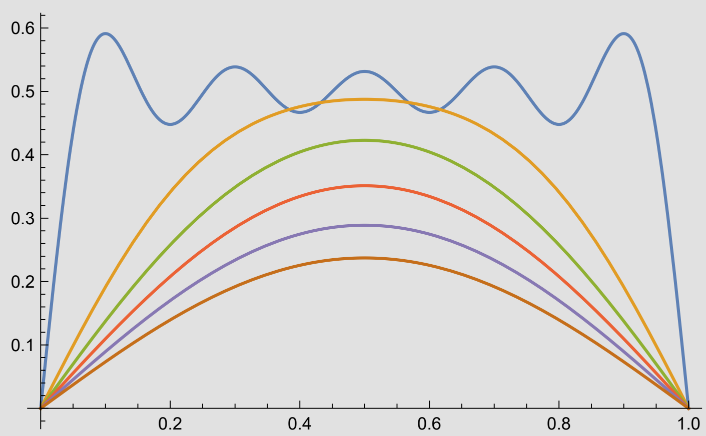
Out[6]=

$$-\frac{\left(-1+(-1)^{n}\right) L}{n \pi}$$

In[7]:=
```
u[x_, t_, nt_, k_, L_] := Sum[A[n, L] Sin[π n x / L] Exp[-k (π n / L)^2 t], {n, 1, nt}]
```

In[8]:=
```
Plot[Evaluate[Table[u[x, t, 10, 1, 1], {t, 0, 0.1, 0.02}]], {x, 0, 1}]
```

Out[8]=



Project

From p.728:12. Three-dimensional plots are constructed with Plot3D. Comment the results you obtain. Interpret the physical significance of the result, making use of the exercises solved above.