Practice Final Examination

Solve the following problems (5 course points each). Present a brief motivation of your method of solution. Problems 9 and 10 are optional; attempt them if you wis to improve your midterm examination score.

1. Solve the eigenvalue problem

$$y'' + 3y' + 2y + \lambda y = 0, y(0) = 0, y(1) = 0.$$

Solution. Linear, second order, constant coefficient, homogeneous ODE with homogeneous boundary conditions (BCs), i.e., an eigenvalue problem. Try solutions of form $y(x) \sim e^{rx}$ to obtain characteristic equation

$$r^2 + 3r + (\lambda + 2) = 0,$$

with solutions

$$r_{1,2} = \frac{1}{2}\left(-3 \pm \sqrt{9 - 4(\lambda + 2)}\right) = -\frac{3}{2} \pm \frac{\sqrt{1 - 4\lambda}}{2} = -\alpha \pm \beta,$$

leading to

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}.$$

For $\beta = 0$, obtain $y(x) = Ae^{-\alpha x} + Bxe^{-\alpha x}$. Apply BCs

$$\begin{array}{ll} x = 0: & A = 0 \\ x = 1: & Ae^{-\alpha} + Be^{-\alpha} = 0 \end{array} \Rightarrow A = B = 0,$$

only a trivial solution.

For $\beta \neq 0$, apply BCs

$$\begin{array}{ll} x = 0: & A + B = 0 \\ x = 1: & Ae^{r_1} + Be^{r_2} = 0 \end{array}.$$

Non-trivial solutions (i.e., $A \neq 0$ or $B \neq 0$) obtained only if principal determinant of above is zero

$$e^{r_2} - e^{r_1} = 0 \Rightarrow e^{-\alpha}(e^{\beta} - e^{-\beta}) = 0.$$

Since $\alpha = -3/2$, $e^{\alpha} \neq 0$, hence

$$e^{\beta} - e^{-\beta} = 2\sinh\beta = 0,$$

with solutions only for $\Delta = 1 - 4\lambda < 0$ in which case $\beta = \frac{i}{2}\sqrt{4\lambda - 1}$, and

$$\sinh\beta = i\sin\frac{\sqrt{4\lambda - 1}}{2} = 0 \Rightarrow \frac{\sqrt{4\lambda - 1}}{2} = k\pi \Rightarrow \lambda_k = \frac{1}{4}[(2k\pi)^2 + 1],$$

eigenvalues of the problem, with associated eigenfunctions

$$y_k(x) = e^{-3x/2}\sin(k\pi x).$$

Note: recall that eigenfunctions are determined up to a multiplicative constant.

2. Solve the eigenvalue problem

$$y'' + \lambda y = 0, y(0) + y'(0) = 0, y(1) + 3y'(1) = 0.$$

Solution. Linear, second order, constant coefficient, homogeneous ODE with homogeneous boundary conditions (BCs), i.e., an eigenvalue problem. Try solutions of form $y(x) \sim e^{rx}$ to obtain characteristic equation $r^2 + \lambda = 0$, with solutions with solutions $r_{1,2} = \pm\sqrt{\lambda}$. When $\lambda = 0$, $y(x) = A + Bx$ and BCs give only the trivial solution $y = 0$. For $\lambda \neq 0$, $y(x) = Ae^{\alpha x} + Be^{-\alpha x}$, $\alpha = \sqrt{\lambda}$, obtain $y'(x) = \alpha(Ae^{\alpha x} - Be^{-\alpha x})$, and BCs give

$$\begin{array}{ll} x = 0: & A + B + \alpha(A - B) = 0 \\ x = 1: & Ae^{\alpha} + Be^{-\alpha} + 3\alpha(Ae^{\alpha} - Be^{-\alpha}) = 0 \end{array} \Rightarrow \left\{ \begin{array}{lllll} (1 + \alpha)A & + & (1 - \alpha)B & = & 0 \\ (1 + 3\alpha)e^{\alpha}A & + & (1 - 3\alpha)e^{-\alpha}B & = & 0 \end{array} \right. .$$

Non-trivial solution obtained if

$$(1-2\alpha-3\alpha^2)e^{-\alpha}-(1+2\alpha-3\alpha^2)e^{\alpha}=0\Rightarrow -2\,(1-3\alpha^2)\sinh\alpha-4\alpha\cosh\alpha=0\Rightarrow \tanh\alpha=-\frac{2\alpha}{1-3\alpha^2}.$$

Sturm-Liouville theorem guarantees existence of a countably infinite number of eigenvalues, impossible for $\alpha\in\mathbb{R}$, thus implying $\alpha=i\beta$ $(\lambda<0)$, $\beta\in\mathbb{R}$, in which case eigenvalues β_k are solutions of

$$\tan\beta=-\frac{2\beta}{1+3\beta^2},$$

with associated ODE solution

$$y_k(x)=Ae^{i\beta x}+Be^{-i\beta x}=(A+B)\cos(\beta x)+i(A-B)\sin(\beta x).$$

Use $x=0$ BC, $A+B=-\alpha(A-B)$ to obtain $A+B=-i\beta(A-B)$, and the eigenfunctions are

$$y_k(x)=\beta_k\cos(\beta_k x)-\sin(\beta_k x)\,.$$

3. Find the Fourier series

$$F(x)=a_0+\sum_{k=1}^{\infty}\left(a_k\cos(kx)+b_k\sin(kx)\right)$$

of $f\colon[0,\pi]\to\mathbb{R}$, $f(x)=2x-3x^2$.

Solution. The system $\{1,\cos x,\sin x,\cos 2x,\sin 2x,\ldots\}$ is an orthogonal basis. Take scalar products

$$\int_0^{\pi}F(x)\cdot 1\,\mathrm{d}x=\pi a_0=\int_0^{\pi}(2x-3x^2)\cdot 1\,\mathrm{d}x=\pi^2-\pi^3=\pi^2(1-\pi)\Rightarrow a_0=\pi(1-\pi)$$

$$\int_0^{\pi}F(x)\cdot\cos(kx)\,\mathrm{d}x=a_k\int_0^{\pi}\cos^2(kx)\,\mathrm{d}x=\frac{a_k\pi}{2}=\int_0^{\pi}(2x-3x^2)\cdot\cos(kx)\,\mathrm{d}x\Rightarrow a_k=\frac{2}{\pi}\int_0^{\pi}(2x-3x^2)\cdot\cos(kx)\,\mathrm{d}x$$

$$b_k=\frac{2}{\pi}\int_0^{\pi}(2x-3x^2)\cdot\sin(kx)\,\mathrm{d}x.$$

Form

$$c_k=\frac{\pi}{2}(a_k+ib_k)=\int_0^{\pi}(2x-3x^2)e^{ikx}\,\mathrm{d}x.$$

Integrate by parts, $u=2x-3x^2$, $\mathrm{d}v=e^{ikx}\,\mathrm{d}x\Rightarrow v=e^{ikx}/(ik)$

$$\int_0^{\pi}(2x-3x^2)e^{ikx}\,\mathrm{d}x=\left[(2x-3x^2)\frac{e^{ikx}}{ik}\right]_{x=0}^{x=\pi}-\frac{1}{ik}\int_0^{\pi}(2-6x)e^{ikx}\,\mathrm{d}x=\frac{3\pi^2-2\pi}{ik}-\frac{1}{ik}\int_0^{\pi}(2-6x)e^{ikx}\,\mathrm{d}x.$$

Another integration by parts, $u=2-6x$, $\mathrm{d}v=e^{ikx}\,\mathrm{d}x\Rightarrow v=e^{ikx}/(ik)$ gives

$$\int_0^{\pi}(2-6x)e^{ikx}\,\mathrm{d}x=\left[(2-6x)\frac{e^{ikx}}{ik}\right]_{x=0}^{x=\pi}+\frac{6}{ik}\int_0^{\pi}e^{ikx}\,\mathrm{d}x=\frac{6\pi-2}{ik}-\frac{2}{ik}+\frac{12}{k^2}=\frac{6\pi-4}{ik}+\frac{12}{k^2}.$$

Obtain

$$c_k=\frac{3\pi^2-2\pi}{ik}-\frac{1}{ik}\left(\frac{6\pi-4}{ik}+\frac{12}{k^2}\right)=\frac{3\pi^2-2\pi}{ik}-\frac{6\pi-4}{k^2}-\frac{12}{ik^3}=\frac{6\pi-4}{k^2}+i\left(\frac{12}{k^3}-\frac{3\pi^2-2\pi}{k}\right)\Rightarrow$$

$$a_k=\frac{2(6\pi-4)}{\pi k^2},\ b_k=\frac{2}{\pi}\left(\frac{12}{k^3}-\frac{3\pi^2-2\pi}{k}\right).$$

4. Find $u(x,t)$, $u\colon[0,1]\times[0,\infty)\to\mathbb{R}$ by solving the problem

$$u_t=u_{xx},\ u(0,t)=0,\ u(1,t)=0,\ u(x,0)=x(1-x).$$

Solution. Linear, second order, homogeneous PDE with homogeneous BCs, inhomogeneous initial condition (IC). Separation of variables $u(x,t)=X(x)T(t)$ leads to

$$\frac{T'}{T}=\frac{X''}{X}=-(n\pi)^2,$$

and solution given as superposition of eigenfunctions of the x-BVP

$$u(x,t)=\sum_{n=1}^{\infty} c_n \sin(n\pi) e^{-(n\pi)^2 t}.$$

Initial condition gives

$$c_n=\int_0^1 x(1-x)\sin(n\pi x)\,\mathrm{d}x.$$

Integration by parts $u=x(1-x)$, $\mathrm{d}v=\sin(n\pi x)\,\mathrm{d}x \Rightarrow v=-\cos(n\pi x)/(n\pi)$

$$c_n=-\left[\frac{x(1-x)\cos(n\pi x)}{n\pi}\right]_{x=0}^{x=1}+\frac{1}{n\pi}\int_0^1(1-2x)\cos(n\pi x)\,\mathrm{d}x.$$

Again, $u=(1-2x)$, $\mathrm{d}v=\cos(n\pi x) \Rightarrow v=\sin(n\pi x)/(n\pi)$

$$\int_0^1(1-2x)\cos(n\pi x)\,\mathrm{d}x=\left[(1-2x)\frac{\sin(n\pi x)}{n\pi}\right]_{x=0}^{x=1}+\frac{2}{n\pi}\int_0^1\sin(n\pi x)\,\mathrm{d}x=-\frac{2}{(n\pi)^2}[\cos(n\pi x)]_{x=0}^{x=1}.$$

Deduce

$$c_{2k}=0,\, c_{2k+1}=\frac{4}{((2k+1)\,\pi)^3}.$$

5. For $z=x+iy$, $\mathrm{Re}(z)>0$ show that

$$\mathrm{Ln}\, z=\frac{1}{2}\log_e(x^2+y^2)+i\tan^{-1}\frac{y}{x},$$

and verify that $\mathrm{Ln}\, z$ thus defined is analytic in the right half-plane.

Solution. From $f(z)=\ln z$, $z=re^{i\theta}$, $r=(x^2+y^2)^{1/2}$, $\theta=\tan^{-1}(y/x)$,

$$f(z)=\ln r+i(\theta+2k\pi).$$

Choose principal branch $k=0$ and obtain above relation. Verify that $\mathrm{Ln}\, z=u+iv$ is analytic by Caucy-Riemann conditions

$$u_x=\frac{x}{x^2+y^2},\, v_y=\frac{(1/x)}{1+(y/x)^2}=\frac{x}{x^2+y^2}=u_x\,\checkmark$$

$$u_y=\frac{y}{x^2+y^2},\, v_x=\frac{-(y/x^2)}{1+(y/x)^2}=-\frac{y}{x^2+y^2}=-u_y\,\checkmark$$

for all points in the right half plane.

6. Show that the real and imaginary parts of $\mathrm{Ln}\, z$ defined above are harmonic.

Solution. As above.

7. Determine the value of the integral

$$I=\int_{1-i}^{1+2i} z e^{z^2}\,\mathrm{d}z.$$

Solution. Integrand has primitive $F(z)=e^{z^2}/2$ hence

$$I=\frac{1}{2}[e^{z^2}]_{z=1-i}^{z=1+2i}=\frac{1}{2}(e^{1+4i-4}-e^{1-2i+1})=\frac{1}{2}(e^{4i-3}-e^{2-2i}).$$

8. Find the value of

$$I = \oint_C \frac{\mathrm{d}z}{z^2(z^2+1)}, C\colon |z-i| = \frac{3}{2}.$$

Solution. Integrand $f(z)$ has simple poles at $z_{1,2} = \pm i$, and a double pole at $z_3 = 0$. The pole $z_1 = -i$ is outside the contour. Apply residue formula

$$I = 2\pi i[\mathrm{res}(f, i) + \mathrm{res}(f, 0)].$$

Compute

$$\mathrm{res}(f, i) = \left[(z-i)\frac{1}{z^2(z^2+1)}\right]_{z=i} = \left[\frac{1}{z^2(z+i)}\right]_{z=i} = -\frac{1}{2i} = \frac{i}{2}.$$

$$\mathrm{res}(f, i) = \left[\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\frac{1}{z^2(z^2+1)}\right)\right]_{z=0} = -\left[\frac{2z}{(z^2+1)^2}\right]_{z=0} = 0.$$

Deduce $I = \pi i$.

9. An elastic cylinder of radius $R = 1$ is subjected to surface force $f(\theta, t) = \cos\theta \sin(\omega t)$. Formulate the wave equation problem for radial displacements $u(\theta, t)$ of the cylinder surface from its equilibrium position.

Solution. Wave equation $u_{tt} = c^2 \nabla \cdot (\nabla u) + f$ in polar coordinates (r, θ) gives

$$u_{tt} = c^2 \nabla \cdot \left[\frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta\right] = \frac{c^2}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} r\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\right] = \frac{c^2}{r^2} u_{\theta\theta} + f.$$

On $r = R = 1$ obtain, $u_{tt} = c^2 u_{\theta\theta} + f$, with periodic BCs $u(0, t) = u(2\pi, t)$, $u_\theta(0, t) = u_\theta(2\pi, t)$ and initial conditions $u(\theta, 0) = 0$, $u_t(\theta, 0) = 0$.

10. Solve the above problem by the separation of variables $u(\theta, t) = \Theta(\theta)\, T(t)$.

Solution. The above is a second-order inhomogeneous PDE. Solve by expanding both $u(\theta, t)$ and $f(\theta, t)$ on the eigenfunctions of the homogeneous PDE $\left\{\frac{1}{2}, \cos\theta, \sin\theta, ..., \cos(n\theta), \sin(n\theta), ...\right\}$

$$u(\theta, t) = \frac{1}{2}a_0(t) + \sum_{n=1}^{\infty} [a_n(t)\cos(n\theta) + b_n(t)\sin(n\theta)], f(\theta, t) = \sin(\omega t)\cos\theta \Rightarrow$$

$$u_{tt} = \frac{1}{2}a_0''(t) + \sum_{n=1}^{\infty} [a_n''(t)\cos(n\theta) + b_n''(t)\sin(n\theta)],$$

$$u_{\theta\theta} = -\sum_{n=1}^{\infty} n^2[a_n(t)\cos(n\theta) + b_n(t)\sin(n\theta)]$$

Since the eigenfunctions are orthogonal obtain ODE system

$$a_0'' = 0, a_1'' = -c^2 a_1 + \sin(\omega t),$$

$$a_n'' = -c^2 a_n, b_n'' = -c^2 b_n, n > 1.$$

Applying initial conditions leads to only one none-zero term, $a_1(t)$.