Mid-term Practice Examination

Answer the following problems (2 course points each). Present brief motivation of your solutions.

Motivation: *Why do power lines hum?*

A rod at rest and initial temperature $u(x,0)=0$ (K), of length $L=\pi$ (m), specific heat capacity $c=1$ (J/kg/K), and thermal diffusivity $k=1$ (m^2/s) has thermally insulated ends held at fixed spatial positions. An electric current of intensity $j(t)=\sin(\omega t)$ is switched on at time $t=0$ (s) and dissipates heat throughout the rod according to Joule's law $q(t)=r\,j^2(t)$ (K/s). The rod experiences linear thermal dilatation leading to longitudinal displacements $\mathrm{d}w=\gamma\,\mathrm{d}u$ (m). In accordance with Hooke's law, thermal dilatation produces a local acceleration $f(x,t)=\gamma\,(u_{tt})_x$, where u_{tt} is the second time derivative of the temperature.

Fundamental physical units: m - meter, s - second, kg - kilogram, K - Kelvin
Derived physical units: J = kg m^2/s - Joule, W=J/s - Watt
Unforced heat equation $u_t=k\nabla^2 u$, Unforced wave equation $w_{tt}=c^2\nabla^2 w$
Alternating current frequency $\nu=60\,\text{Hz}$, $\omega=2\pi\nu$.
In Octave 1, B-flat frequency: 117 Hz, B frequency: 123 Hz. 1 Hz = 1/s.

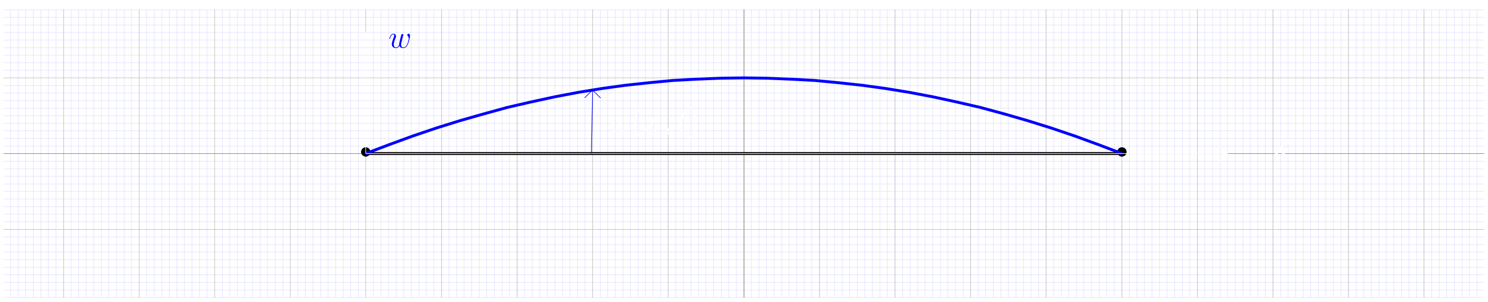

Figure 1. Rod thermal transfer and wave motion

1. Write the initial, boundary value problem (IBVP) describing temperature evolution in the rod. Classify the resulting problem.

   Solution. This is a forced heat conduction problem. Thermally insulated implies no heat flux at ends, hence homogeneous Neumann conditions.

   $$u_t=ku_{xx}+q,\, t>0,\, 0<x<L\,(\text{forced heat PDE})$$
   $$u_x(0,t)=0,\, u_x(L,t)=0\,(\text{Boundary conditions})$$
   $$u(x,0)=0\,(\text{Initial condition}).$$

   The above is a linear, second-order, inhomogeneous PDE with homogeneous boundary and initial conditions.

2. Write the IBVP describing longitudinal elastic wave propagation in the rod. Classify the resulting problem.

   Solution. This is a forced wave propagation problem. Fixed ends implies homogeneous Dirichlet conditions.

   $$w_t=c^2\,w_{xx}+f,\, t>0,\, 0<x<L\,(\text{forced heat PDE})$$
   $$w(0,t)=0,\, w(L,t)=0\,(\text{Boundary conditions})$$
   $$w(x,0)=0,\, w_t(x,0)=0\,(\text{Initial condition}).$$

   The above is a linear, second-order, inhomogeneous PDE with homogeneous boundary and initial conditions.

3. Solve the heat IBVP.

Solution. Express $u(x,t)$, $q(t)$ as series in the eigenfunctions $y_n(x) = \cos(nx)$ of the homogeneous Neumann condition Sturm-Liouville problem

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos(nx),\; q(t) = r\sin^2(\omega t) = r\sin^2(\omega t)\cdot\cos(0\cdot x).$$

Obtain the ODEs for $a_n(t)$

$$\dot{a}_0(t) = r\sin^2(\omega t) = \frac{r}{2}[1-\cos(2\omega t)] \Rightarrow a_0(t) = \frac{r}{2}\left[t - \frac{1}{2\omega}\sin(2\omega t)\right]$$

$$\dot{a}_n(t) = -kn^2\, a_n(t) \Rightarrow a_n(t) = A e^{-kn^2 t},\, a_n(0) = 0 \Rightarrow A = 0$$

Solution is

$$u(x,t) = \frac{r}{2}\left[t - \frac{1}{2\omega}\sin(2\omega t)\right] + \sum_{n=1}^{\infty} (e^{-kn^2 t} - 1)\cos(nx).$$

4. Solve the elastic wave IBVP.

Solution. Use the above heat equation solution to obtain the forcing term

$$f(x,t) = \gamma\,(u_{tt})_x = \gamma n\,(1 - k^2 n^4 e^{-kn^2 t})\sin(nx).$$

Express $w(x,t)$ and $f(x,t)$ as series in the eigenfunctions $z_n(x) = \sin(nx)$ of the homogeneous Dirichlet condition Sturm-Liouville problem

$$w(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx),\; f(x,t) = \sum_{n=1}^{\infty} d_n(t)\sin(nx).$$

Obtain the forcing coefficients

$$d_n(t) = \gamma n\,(1 - k^2 n^4 e^{-kn^2 t}).$$

Obtain coefficient ODEs

$$\ddot{c}_n = -(nc)^2 c_n + d_n.$$

The homogeneous ODE solutions are

$$c_n^{(h)}(t) = A_n\cos(nct) + B_n\sin(nct).$$

The inhomogeneous particular ODE solution suggested by the d_n forcing term is

$$c_n(t) = c_n^{(h)}(t) + C_n + D_n e^{-kn^2 t}.$$

Replacing,

$$k^2 n^4 D_n e^{-kn^2 t} = -(nc)^2(C_n + D_n e^{-kn^2 t}) + \gamma n\,(1 - k^2 n^4 e^{-kn^2 t}).$$

Identify coefficients of independent functions 1, $e^{-kn^2 t}$

$$0 = -(nc)^2 C_n + \gamma n,\, k^2 n^4 D_n = -(nc)^2 D_n - \gamma n k^2 n^4 \Rightarrow$$

$$C_n = \frac{\gamma}{nc^2},\, D_n = -\frac{\gamma n k^2 n^4}{k^2 n^4 + (nc)^2} = \frac{\gamma k^2 n^3}{k^2 n^2 + c^2}.$$

Obtain solution

$$w(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx),\, c_n(t) = A_n\cos(nct) + B_n\sin(nct) + \frac{nc^2}{\gamma} + \frac{\gamma k^2 n^3}{k^2 n^2 - c^2}\, e^{-kn^2 t}.$$

Initial conditions specify A_n, B_n

$$c_n(0) = 0 = A_n + \frac{n c^2}{\gamma} + \frac{\gamma k^2 n^3}{k^2 n^2 - c^2} \Rightarrow A_n = -\frac{n c^2}{\gamma} - \frac{\gamma k^2 n^3}{k^2 n^2 - c^2},$$

$$\dot{c}_n(0) = 0 = n c B_n - \frac{\gamma k^3 n^5}{k^2 n^2 - c^2} \Rightarrow B_n = \frac{\gamma k^3 n^4}{c\,(k^2 n^2 - c^2)}$$

5. It has been observed that power lines hum at a pitch between B-flat and B. Propose an explanation.

   Solution. The thermal forcing term $\sin^2(\omega t)$ leads excitation at the double frequency 2ω in accordance with the trigonometric identity

$$\sin^2(\omega t) = 1 - \cos(2\omega t).$$

   Since the AC frequency is $\nu = 60\,\mathrm{Hz}$, a sound at $2\nu \cong 120\,\mathrm{Hz}$ is heard.