Final Examination

Solve the following problems (4 course points each). Present a brief motivation of your method of solution. Answers without explanation of solution procedure are not awarded credit.

1. Use the Laplace transform $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ to solve the problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, x > 0, t > 0,$$

$$u(0,t) = 0, \lim_{x\to\infty} \frac{\partial u}{\partial x}(x,t) = 0, t > 0,$$

$$u(x,0) = 0, v(x,0) = \frac{\partial u}{\partial t}(x,0) = -v_0, x > 0, v \in \mathbb{R}_+.$$

2. Use the Fourier transform $F(\alpha) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,\mathrm{d}x$ to solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < \pi, y > 0,$$

$$u(0,y) = f(y), \frac{\partial u}{\partial x}(\pi, y) = 0, y > 0,$$

$$\frac{\partial u}{\partial y}(x,0) = 0, 0 < x < \pi.$$

3. Find all solutions of the equation $z^8 - 2z^4 + 1 = 0$. Write the roots in both Cartesian and polar form.
4. Sketch the region defined by $-1 \leqslant \mathrm{Im}(1/z) < 1$. Is this region a domain?
5. Is $f(z) = x^2 - x + y + i(y^2 - 5y - x)$ an analytic function? Is it differentiable along the curve $y = x + 2$?
6. Evaluate the integral

$$\oint_C \frac{2z}{z^2+3}\,\mathrm{d}z$$

   for C defined as:

   a) $|z| = 1$;

   b) $|z - 2i| = 1$.

7. Expand

$$f(z) = \frac{1}{z(1-z)^2}$$

   in a Laurent series valid for:

   a) $0 < |z| < 1$;

   b) $|z| > 1$.