Final Examination

Solve the following problems (4 course points each). Present a brief motivation of your method of solution. Answers without explanation of solution procedure are not awarded credit.

1. Use the Laplace transform $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ to solve the problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, x>0, t>0,$$

$$u(0,t) = 0, \lim_{x\to\infty} \frac{\partial u}{\partial x}(x,t) = 0, t>0,$$

$$u(x,0) = 0, v(x,0) = \frac{\partial u}{\partial t}(x,0) = -v_0, x>0, v\in\mathbb{R}_+ .$$

Solution. With notation $u_t = \partial u/\partial t$, $u_{tt} = \partial^2 u/\partial t^2$, use integration by parts to evaluate

$$\mathcal{L}\{u'(x,t)\} = \int_0^\infty e^{-st} u_t(x,t)\,\mathrm{d}t = [e^{-st} u(x,t)]_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st} u(x,t)\mathrm{d}t = -u(x,0) + sU(x,s)$$

$$\mathcal{L}\{u'(x,t)\} = \int_0^\infty e^{-st} u_{tt}(x,t)\,\mathrm{d}t = [e^{-st} u_t(x,t)]_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st} u_t(x,t)\mathrm{d}t = -u_t(x,0) + s[sU(x,s) - u(x,0)].$$

Apply $\mathcal{L}$ to PDE to obtain

$$a^2 \frac{\mathrm{d}^2 U(x,s)}{\mathrm{d}x^2} = s^2 U(x,s) - s\,u(x,0) - u_t(x,0).$$

Apply initial conditions to obtain

$$a^2 \frac{\mathrm{d}^2 U(x,s)}{\mathrm{d}x^2} = s^2 U(x,s) + v_0.$$

Solve the homogeneous ODE

$$a^2 \frac{\mathrm{d}^2 U(x,s)}{\mathrm{d}x^2} = s^2 U(x,s),$$

to obtain the general solution

$$U(x,s) = c_1 e^{-sx/a} + c_2 e^{sx/a}.$$

A particular solution is

$$U(x,s) = c_1 e^{-sx/a} + c_2 e^{sx/a} - \frac{v_0}{s^2}.$$

Take the Laplace transform of the boundary conditions:

$$\lim_{x\to\infty} \frac{\mathrm{d}U(x,s)}{\mathrm{d}x} = 0 \Rightarrow c_2 = 0$$

$$U(0,s) = 0 \Rightarrow c_1 - \frac{v_0}{s^2} = 0 \Rightarrow c_1 = \frac{v_0}{s^2},$$

leading to solution

$$U(x,s) = \frac{v_0}{s^2}(e^{-sx/a} - 1).$$

Take the inverse Laplace transforms

$$u(t,x) = \mathcal{L}^{-1}\{U(x,s)\} = v_0\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2} e^{-sx/a}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right].$$

From

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\,\mathrm{d}t = -\left[\frac{1}{s} e^{-st} t\right]_{t=0}^{t\to\infty} + \frac{1}{s}\int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s^2},$$

deduce

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Recall that exponential factors in a Laplace transform arise from a delay that can be represented through the unit step function $H(t) = 1$ for $t > 0$, 0 otherwise, as

$$\mathcal{L}\{H(t-c)\} = \int_0^\infty e^{-st} H(t-c)\ \mathrm{d}t = \int_c^\infty e^{-st}\ \mathrm{d}t = -\frac{1}{s}[e^{-st}]_{t=c}^{t\to\infty} = \frac{e^{-cs}}{s}, (c > 0).$$

To obtain an s^2 denominator, combine above with integration by parts

$$\mathcal{L}\{t H(t-c)\} = \int_c^\infty t e^{-st}\ \mathrm{d}t = -\frac{1}{s}[t e^{-st}]_{t=c}^{t\to\infty} + \frac{1}{s}\int_c^\infty e^{-st}\ \mathrm{d}t = \frac{c e^{-cs}}{s} + \frac{e^{-cs}}{s^2} = c\mathcal{L}\{H(t-c)\} + \frac{e^{-cs}}{s^2},$$

to obtain

$$\frac{e^{-cs}}{s^2} = \mathcal{L}\{t H(t-c)\} - c\mathcal{L}\{H(t-c)\} = \mathcal{L}\{(t-c)\, H(t-c)\} \Rightarrow (t-c)\, H(t-c) = \mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s^2}\right\}.$$

Gathering the above results

$$u(t,x) = v_0\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2} e^{-sx/a}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right] = v_0\left[\left(t - \frac{x}{a}\right) H\left(t - \frac{x}{a}\right) - t\right].$$

2. Use the Fourier transform $F(\alpha) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, \mathrm{d}x$ to solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < \pi, y > 0,$$

$$u(0,y) = f(y), \frac{\partial u}{\partial x}(\pi, y) = 0, y > 0,$$

$$\frac{\partial u}{\partial y}(x,0) = 0, 0 < x < \pi.$$

Solution. The problem is defined on the half-infinite strip $y > 0$, and the boundary condition at $y = 0$ is on the derivative of the function, $u_y(x,0) = 0$. The appropriate transform is the cosine transform

$$U(x,\beta) = \mathcal{F}\{u(x,y)\} = \int_0^\infty \cos(\beta y)\, u(x,y)\, \mathrm{d}y.$$

Integration by parts, assuming $u, \partial u / \partial y \to 0$, as $y \to \infty$ leads to

$$\int_0^\infty \cos(\beta y)\, \frac{\partial^2 u}{\partial y^2}(x,y)\, \mathrm{d}y = \left[\cos(\beta y) \frac{\partial u}{\partial y}(x,y)\right]_{y=0}^{y\to\infty} + \beta \int_0^\infty \sin(\beta y)\, \frac{\partial u}{\partial y}(x,y)\, \mathrm{d}y =$$

$$-\frac{\partial u}{\partial y}(x,0) + \beta \int_0^\infty \sin(\beta y)\, \frac{\partial u}{\partial y}(x,y)\, \mathrm{d}y = -\frac{\partial u}{\partial y}(x,0) + \beta\left([\sin(\beta y) u]_{y=0}^{y\to\infty} - \beta \int_0^\infty \cos(\beta y)\, u(x,y)\, \mathrm{d}y \right) =$$

$$-\frac{\partial u}{\partial y}(x,0) - \beta^2 U(x,\beta).$$

Replacing into Laplace PDE and using boundary condition at $y = 0$, gives

$$\frac{\mathrm{d}^2 U}{\mathrm{d}x^2} - \beta^2 U = -\frac{\partial u}{\partial y}(x,0) = 0,$$

with solution

$$U(x,\beta) = c_1(\beta) \cosh(\beta x) + c_2(\beta) \sinh(\beta x).$$

Cosine transform of the boundary condition at $x=0$ leads to

$$U(0,\beta)=F(\beta)=\int_0^\infty \cos(\beta y)\, f(y)\mathrm{d}y=c_1(\beta),$$

specifying that $c_1(\beta)=F(\beta)$, the cosine transform of the boundary value $f(y)$. Compute the x-derivative

$$\frac{\mathrm{d}U}{\mathrm{d}x}(x,\beta)=\beta[c_1(\beta)\sinh(\beta x)+c_2(\beta)\cosh(\beta x)].$$

The boundary condition at $x=\pi$ specifies

$$\frac{\mathrm{d}U}{\mathrm{d}x}(\pi,\beta)=0=\beta[c_1(\beta)\sinh(\beta\pi)+c_2(\beta)\cosh(\beta\pi)]\Rightarrow c_2(\beta)=-c_1(\beta)\frac{\sinh(\beta\pi)}{\cosh(\beta\pi)}=-F(\beta)\tanh(\beta\pi).$$

The solution is obtained by taking the inverse cosine transform of

$$U(x,\beta)=F(\beta)[\cosh(\beta x)-\tanh(\beta\pi)\sinh(\beta x)]\Rightarrow$$

$$u(x,y)=\frac{2}{\pi}\int_0^\infty F(\beta)[\cosh(\beta x)-\tanh(\beta\pi)\sinh(\beta x)]\cos(\beta y)\,\mathrm{d}\beta=\frac{2}{\pi}\int_0^\infty F(\beta)\cosh[\beta(x-\pi)]\frac{\cos(\beta y)}{\cos(\beta\pi)}\,\mathrm{d}\beta.$$

3. Find all solutions of the equation $z^8-2z^4+1=0$. Write the roots in both Cartesian and polar form.

   Solution. Introduce $w=z^4$ to obtain $w^2-2w+1=(w-1)^2=0$ with double root $w_{1,2}=1$. The equation $z^4-1=0\Rightarrow z^4=e^{2\pi i}$, has roots $z_k=e^{ik\pi/2}$ for $k=0,1,2,3$. The original equation $z^8-2z^4+1=0$ as double roots at each z_k.

4. Sketch the region defined by $-1\leqslant \operatorname{Im}(1/z)<1$. Is this region a domain?

   Solution. With $z=x+iy$,

$$\operatorname{Im}\left(\frac{1}{x+iy}\right)=\operatorname{Im}\left(\frac{x-iy}{x^2+y^2}\right)=-\frac{y}{x^2+y^2}.$$

The inequality $-y/(x^2+y^2)<1$ leads to $-y<x^2+y^2\Rightarrow x^2+\left(y+\frac{1}{2}\right)^2-\frac{1}{4}>0$, the exterior of a circle of radius $1/2$ centered at $(0,-1/2)$, excluding the circle. The inequality $-1\leqslant -y/(x^2+y^2)$ leads to $x^2+y^2-y\geqslant 0\Rightarrow x^2+\left(y-\frac{1}{2}\right)^2-\frac{1}{4}\geqslant 0$, the exterior of a circle of radius $1/2$, centered at (0,1/2). The region is not a domain since it is not an open set.

5. Is $f(z)=x^2-x+y+i(y^2-5y-x)$ an analytic function? Is it differentiable along the curve $y=x+2$?

   Solution. With $z=x+iy$, $f(z)=u(x,y)+iv(x,y)$,

$$u(x,y)=x^2-x+y, v=y^2-5y-x$$

the first Cauchy-Riemann condition is not verified, i.e.,

$$u_x=2x-1\neq v_y=2y-5, u_y=1=-v_x$$

so f is not analytic. On the curve C: $y=x+2$ the function restriction is a quadratic polynomial in x

$$f_C(x)=x^2-x+x+2+i[(x+2)^2-5(x+2)-x]=x^2+2+i(x^2-2x-6),$$

and is differentiable, $f_C'(x)=2x+2i(x-1)$.

6. Evaluate the integral

$$\oint_C \frac{2z}{z^2+3}\,\mathrm{d}z$$

for C defined as:

a) $|z|=1$;

   Solution. Assume C traversed in positive (CCW) direction. The integrand has isolated pole singularities at $z_{1,2}=\pm i\sqrt{3}$ not within the unit circle, hence

$$\oint_C \frac{2z}{z^2+3}\,\mathrm{d}z=0$$

b) $|z-2i|=1$.

Solution. The curve C is a circle of radius 1 centered at $(0,2)$. The singularity $z_1=i\sqrt{3}$ is within the CCW-traversed contour, the singularity $z_2=-i\sqrt{3}$ is outside. The integral is therefore

$$\oint_C \frac{2z}{z^2+3}\,\mathrm{d}z=\oint_C f(z)\,\mathrm{d}z=2\pi i\,\mathrm{Res}[f(z=i\sqrt{3})].$$

From

$$\frac{2z}{z^2+3}=\frac{1}{z-i\sqrt{3}}+\frac{1}{z+i\sqrt{3}}$$

observe $\mathrm{Res}[f(z=i\sqrt{3})]=1$.

7. Expand

$$f(z)=\frac{1}{z(1-z)^2}$$

in a Laurent series valid for:

a) $0<|z|<1$;

Solution. Differentiation term-by-term of the convergent series for $|z|<1$

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots\Rightarrow\frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+\cdots$$

and multiplying by $1/z$ ($|z|>0\Rightarrow z\neq 0$) gives

$$f(z)=\frac{1}{z}+2+3z+4z^2+5z^3+\cdots+(k+2)z^k+\cdots$$

b) $|z|>1$.

Solution. Substitute $w=1/z$, $|w|<1$, and use above series to obtain

$$\frac{1}{z(1-z)^2}=\frac{w}{\left(1-\frac{1}{w}\right)^2}=\frac{w^3}{(w-1)^2}=\frac{w^3}{(1-w)^2}=w^3+2w^4+3w^5+\cdots.$$

Substitue back $z=1/w$ and deduce

$$f(z)=\frac{1}{z^3}+\frac{2}{z^4}+\frac{3}{z^5}+\cdots+\frac{k-2}{z^k}+\cdots.$$