Overview

- Orthogonal functions
- Fourier Series
- Fourier Cosine and Sine Series

Definition. *In* $\mathbb{R}^3$ *the inner product* $(\boldsymbol{u},\boldsymbol{v})$ *of two vectors* $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^3$ *is defined as*

$$\boldsymbol{u}=\begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix}, \boldsymbol{v}=\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}, (\boldsymbol{u},\boldsymbol{v})=u_1v_1+u_2v_2+u_3v_3=\sum_{i=1}^{3} u_i\,v_i\cdot 1$$

Definition. *In general a scalar product* $(\,,\,)\colon\mathcal{U}\times\mathcal{U}\to\mathbb{R}$ *has properties:*

1. $(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{v},\boldsymbol{u})$
2. $(k\boldsymbol{u}+l\boldsymbol{v},\boldsymbol{w})=k\,(\boldsymbol{u},\boldsymbol{w})+l\,(\boldsymbol{v},\boldsymbol{w})$
3. $(\boldsymbol{u},\boldsymbol{u})=0$ *if* $\boldsymbol{u}=\boldsymbol{0}$, *and* $(\boldsymbol{u},\boldsymbol{u})>0$ *if* $\boldsymbol{u}\neq\boldsymbol{0}$

Definition. *The inner product of two functions* $f,g\colon[a,b]\to\mathbb{R}$ *is defined as*

$$(f,g)=\int_a^b f(x)\,g(x)\,\mathrm{d}x$$

Example. The scalar product of $f(x)=x$, $g(x)=x^2$, $f,g\colon[-1,1]\to\mathbb{R}$.

$$(f,g)=\int_{-1}^{1} x\cdot x^2\,\mathrm{d}x=\frac{1}{4}x^4\Big|_{x=-1}^{x=1}=0$$

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In[6]:= ScProd[f_,g_]:=Integrate[f g,{x,-1,1}];
        Table[ScProd[x^l,x^m],{l,1,4},{m,1,4}]
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$$\begin{pmatrix} \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} \\ \frac{2}{5} & 0 & \frac{2}{7} & 0 \\ 0 & \frac{2}{7} & 0 & \frac{2}{9} \end{pmatrix}$$

Definition. *Functions f,g are orthogonal if their scalar product is null, $(f,g)=0$.*

Definition. *Functions $\{\phi_1,\phi_2,\ldots\}$ are an orthogonal set if $\forall i\neq j\Rightarrow(\phi_i,\phi_j)=0$.*

- The 2-norm of vector $\boldsymbol{u} \in \mathbb{R}^n$ given by inner product $\|\boldsymbol{u}\| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$.
- Similarly for functions, $\| f \| = (f, f)^{1/2}$

Definition. *Functions* $\{\phi_1, \phi_2, \ldots\}$ *are an orthonormal set if* $\forall i \neq j \Rightarrow (\phi_i, \phi_j) = 0$, *and* $\forall i$, $(\phi_i, \phi_i) = 1$. *Using Kronecker delta* $(\phi_i, \phi_j) = \delta_{ij}$.

- $\left\{ \frac{1}{\sqrt{\pi}} \cdot 1, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \cos 2x, \ldots \right\}$ is an orthogonal set on $[-\pi, \pi]$

$$(\cos mx, \cos nx) = \int_{-\pi}^{\pi} (\cos mx)(\cos nx)\, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x + \cos(m+n)x]\, \mathrm{d}x = \pi \delta_{mn}$$

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In[2]:= ScCos[m_,n_]:=Integrate[Cos[m x] Cos[n x],{x,-Pi,Pi}];
        Table[ScCos[m,n],{m,1,3},{n,1,3}]
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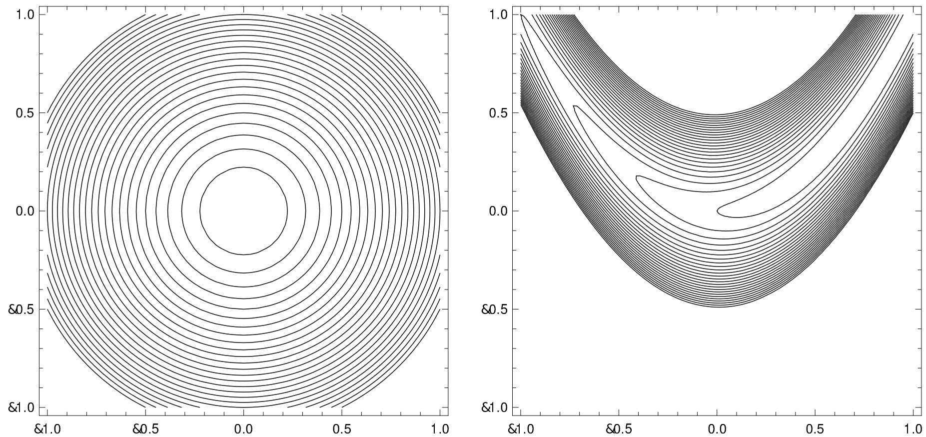
$$\begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$$

- For $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^2$, $\boldsymbol{u}\cdot\boldsymbol{v}=u_1v_1+u_2v_2$, $\boldsymbol{u}=\left[\begin{array}{c} u_1 \\ u_2 \end{array}\right], \boldsymbol{v}=\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right], \boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{u}^T\boldsymbol{v}=\boldsymbol{u}^T\boldsymbol{I}\boldsymbol{v}$
- The inner product reflects Euclidean geometry, other geometries described by a weighted version of the inner product $(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{A}}=\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{v}$, with $\boldsymbol{A}$ s.p.d



- Similarly for functions $f,g\colon[a,b]\to\mathbb{R}$, weighted scalar products are defined as

$$(f,g)=\int_a^b w(x)\,f(x)\,g(x)\,\mathrm{d}x$$

- Find the expansion of $f(x)$ on the set $\{\phi_1, \phi_2, \phi_3, \ldots\}$, e.g.:
  - expansion on the cosines $\{1, \cos x, \cos 2x, \ldots\}$
  - expansion on the sines $\{\sin x, \sin 2x, \ldots\}$
  - expansion on the trigonometric basis $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$ that is analogous to the expansion of a vector

$$\boldsymbol{u} = u_1 \boldsymbol{e}_1 + u_2 \boldsymbol{e}_2 + u_3 \boldsymbol{e}_3, u_1 = (\boldsymbol{u}, \boldsymbol{e}_1), u_2 = (\boldsymbol{u}, \boldsymbol{e}_2), u_3 = (\boldsymbol{u}, \boldsymbol{e}_3)$$

- Find a trigonometric (Fourier) series for $f: [-\pi, \pi] \to \mathbb{R}$, $f(x) = f(x + 2\pi)$

$$f(x) = \frac{1}{2} A_0 \cdot 1 + A_1 \cdot \cos x + B_1 \cdot \sin x + A_2 \cdot \cos 2x + B_2 \cdot \sin 2x + \ldots$$

- The coefficients are

$$A_0 = \frac{1}{\pi}(f, 1), A_1 = \frac{1}{\pi}(f, \cos x), B_1 = \frac{1}{\pi}(f, \sin x),$$

- In general for $f: [-p, p] \to \mathbb{R}$

$$A_k = \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(k \frac{\pi}{p} x\right), B_k = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(k \frac{\pi}{p} x\right).$$

Fourier series convergence

- For $f\colon[-p,p]\to\mathbb{R}$, f, f' piecewise continuous the Fourier series converges to $f(x)$ except at discontinuities where it converges to the arithmetic average

$$\frac{1}{2}A_0+\sum_{k=1}^{\infty}\left(A_k\cos\left[k\,\frac{\pi}{p}\,x\right]+B_k\sin\left[k\,\frac{\pi}{p}\,x\right]\right)\to\frac{1}{2}[f(x_-)+f(x_+)]$$

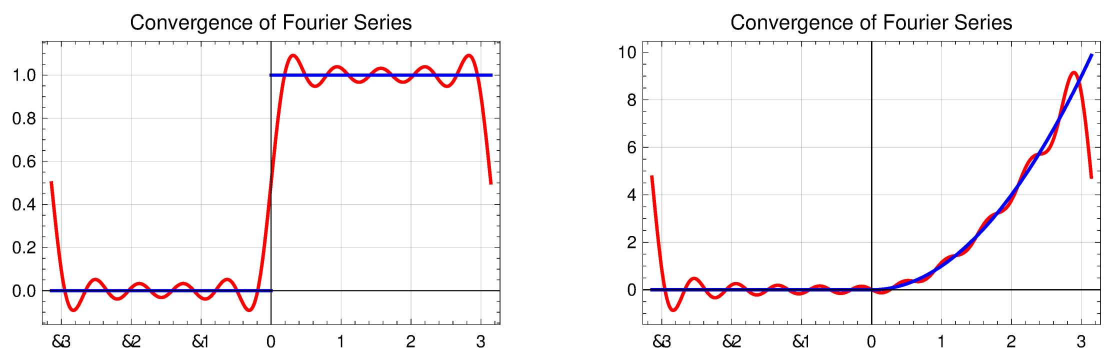


Figure 1. Convergence behavior of Fourier series of discontinuous functions

- Consider $f\colon [-p,p] \to \mathbb{R}$, $f(x) = f(-x)$, i.e., f is an even function. Then

$$f(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos\left[k\,\frac{\pi}{p}\,x\right]$$

for f, f' piecewise continuous

- Consider $f\colon[-p,p]\to\mathbb{R}$, $f(x)=-f(-x)$, i.e., f is an odd function. Then

$$f(x)=\sum_{k=1}^{\infty} B_k \sin\left[k\,\frac{\pi}{p}\,x\right]$$

for f, f' piecewise continuous