



## Overview

- Complex Fourier series
- Sturm-Liouville problem
- Orthogonal function series



- For  $f: [-p, p] \rightarrow \mathbb{C}$ ,  $f, f'$  piecewise continuous

$$f(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} \left( A_k \cos\left[k \frac{\pi}{p} x\right] + B_k \sin\left[k \frac{\pi}{p} x\right] \right),$$
$$A_k = \frac{1}{p} \int_{-p}^p f(x) \cos\left(k \frac{\pi}{p} x\right) \in \mathbb{C}, B_k = \frac{1}{p} \int_{-p}^p f(x) \sin\left(k \frac{\pi}{p} x\right).$$

- Use  $\theta_k = k \frac{\pi}{p} x$ ,  $\cos \theta_k = \frac{1}{2}(e^{i\theta_k} + e^{-i\theta_k})$ ,  $\sin \theta_k = \frac{1}{2i}(e^{i\theta_k} - e^{-i\theta_k})$  to obtain

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{n \frac{i\pi}{p} x},$$
$$C_k = \frac{1}{2}(A_k - i B_k), C_{-k} = \frac{1}{2}(A_k + i B_k), k \in \mathbb{N},$$
$$C_n = \frac{1}{2p} \int_{-p}^p f(x) \exp\left(-\frac{i n \pi}{p} x\right).$$



- Fourier series expresses  $f: [-\pi, \pi]$  as a linear combination of  $\sin(kx), \cos(kx)$ .
- The set  $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots\}$  form an orthogonal basis
- Question: are there other basis sets? Answer: Yes, solutions of Sturm-Liouville
- Regular Sturm-Liouville problem for  $y: [a, b] \rightarrow \mathbb{R}$ ,  $r, q, p, r' \in \mathcal{C}[a, b]$ ,  $\lambda \in \mathbb{R}$ ,

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0,$$

$$A_1 y(a) + B_1 y'(a) = 0, A_2 y(b) + B_2 y'(b) = 0,$$

$$A_1 B_1 \neq 0, A_2 B_2 \neq 0.$$

- Sturm-Liouville scalar product

$$(f, g)_p = \int_a^b p(x) f(x) g(x) dx$$

- The regular Sturm-Liouville problem has non-zero eigensolutions  $(\lambda_n, y_n)$

$\exists \lambda_n \in \mathbb{R}, \lambda_1 < \lambda_2 < \dots, \lambda_n \rightarrow \infty \text{ as } n \rightarrow \infty,$	$\forall \lambda_n, \exists! \text{ solution } y_n,$
$\{y_1, y_2, \dots\}$ linearly independent,	$(y_j, y_k)_p = \delta_{jk}.$



- Sturm-Liouville problem for  $y: [a, b] \rightarrow \mathbb{R}$ ,  $r, q, p, r' \in \mathcal{C}[a, b]$ ,  $r, p > 0$ ,  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned} \frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y &= 0, \\ A_1 y(a) + B_1 y'(a) &= 0, A_2 y(b) + B_2 y'(b) = 0, \\ A_1 B_1 &\neq 0, A_2 B_2 \neq 0. \end{aligned}$$

- Singular Sturm-Liouville problem

$$r(a) = 0 \text{ and } A_1 y(a) + B_1 y'(a) = 0, r(b) = 0 \text{ and } A_2 y(b) + B_2 y'(b) = 0.$$

- Periodic Sturm-Liouville problem

$$r(a) = r(b) = 0 \text{ and no BC's, } r(a) = r(b) \text{ and } y(a) = y(b), y'(a) = y'(b)$$



- For  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$   $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{y}$ . Consider operator  $\mathbf{A} \in \mathbb{R}^{m \times m}$

$$(\mathbf{A}\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{A}^T \mathbf{y} = (\mathbf{x}, \mathbf{A}^T \mathbf{y})$$

If  $\mathbf{A} = \mathbf{A}^T$  (symmetric) then  $(\mathbf{A}\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}\mathbf{y})$ , same operator is either slot.

- Consider now  $u, v: [a, b] \rightarrow \mathbb{R}$ , the Sturm-Liouville operator

$$\mathcal{L}u = \left( \frac{d}{dx} \left[ r(x) \frac{d}{dx} \right] + [q(x) + \lambda p(x)] \right) u$$

satisfies  $(\mathcal{L}u, v)_p = (u, \mathcal{L}v)_p$  and is said to be *self-adjoint*.

- Any ODE  $a(x)y'' + b(x)y' + [c(x) + \lambda d(x)]y = 0$  can be made self-adjoint

$$\frac{d}{dx} [e^{\int (b/a) dx} y'] + \left[ \frac{c}{a} e^{\int (b/a) dx} + \lambda \frac{d}{a} e^{\int (b/a) dx} \right] y = 0$$



- Bessel equation

$$x y'' + y' + \left( a^2 x - \frac{n^2}{x} \right) y = 0 \Leftrightarrow \frac{d}{dx}(x y') + \left( a^2 x - \frac{n^2}{x} \right) y = 0,$$

with solutions  $J_n(ax), Y_n(ax)$ .

- Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \Leftrightarrow \frac{d}{dx}[(1 - x^2)y'] + n(n + 1)y = 0,$$

with solutions  $P_n(x)$ .



- Similar to Fourier series  $f(x)$  can be expressed as a linear combination of other basis sets, e.g., Bessel, Legendre functions

$$f(x) = \sum_{n=1}^{\infty} c_n J(a_n x)$$

$$f(x) = \sum_{n=1}^{\infty} c_n P_n(x)$$