Overview

- Complex Fourier series
- Sturm-Liouville problem
- Orthogonal function series

- For $f\colon [-p,p] \to \mathbb{C}$, f, f' piecewise continuous

$$f(x) = \tfrac{1}{2}A_0 + \sum_{k=1}^{\infty} \left(A_k \cos\left[k\,\frac{\pi}{p}\,x\right] + B_k \sin\left[k\,\frac{\pi}{p}\,x\right]\right),$$

$$A_k = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(k\,\frac{\pi}{p}x\right) \in \mathbb{C},\ B_k = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(k\,\frac{\pi}{p}x\right).$$

- Use $\theta_k = k\,\frac{\pi}{p}x$, $\cos\theta_k = \frac{1}{2}(e^{i\theta_k} + e^{-i\theta_k})$, $\sin\theta_k = \frac{1}{2i}(e^{i\theta_k} - e^{-i\theta_k})$ to obtain

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{n\frac{i\pi}{p}x},$$

$$C_k = \tfrac{1}{2}(A_k - i\,B_k),\ C_{-k} = \tfrac{1}{2}(A_k + i\,B_k),\ k \in \mathbb{N},$$

$$C_n = \frac{1}{2p}\int_{-p}^{p} f(x)\exp\left(-\frac{i\,n\,\pi}{p}x\right).$$

- Fourier series expresses $f\colon[-\pi,\pi]$ as a linear combination of $\sin(kx),\cos(kx)$.
- The set $\{1,\cos(x),\sin(x),\cos(2x),\sin(2x),\dots\}$ form an orthogonal basis
- Question: are there other basis sets? Answer: Yes, solutions of Sturm-Liouville
- Regular Sturm-Liouville problem for $y\colon[a,b]\to\mathbb{R}$, $r,q,p,r'\in\mathcal{C}[a,b]$, $\lambda\in\mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[r(x)y']+[q(x)+\lambda p(x)]\,y=0,$$
$$A_1\,y(a)+B_1\,y'(a)=0,\;A_2\,y(b)+B_2\,y'(b)=0,$$
$$A_1B_1\neq 0,\;A_2B_2\neq 0.$$

- Sturm-Liouville scalar product

$$(f,g)_p=\int_a^b p(x)\,f(x)g(x)\mathrm{d}x$$

- The regular Sturm-Liouville problem has non-zero eigensolutions (λ_n,y_n)

$\exists\,\lambda_n\in\mathbb{R},\lambda_1<\lambda_2<\cdots,\lambda_n\to\infty$ as $n\to\infty$,	$\forall\lambda_n,\exists!$ solution y_n,
$\{y_1,y_2,\dots\}$ linearly independent,	$(y_j,y_k)_p=\delta_{jk}$.

- Sturm-Liouville problem for $y\colon [a, b] \to \mathbb{R}$, $r, q, p, r' \in \mathcal{C}[a, b]$, $r, p > 0$, $\lambda \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[r(x) y'] + [q(x) + \lambda p(x)]\, y = 0,$$
$$A_1\, y(a) + B_1\, y'(a) = 0, A_2\, y(b) + B_2\, y'(b) = 0,$$
$$A_1 B_1 \neq 0, A_2 B_2 \neq 0.$$

- Singular Sturm-Liouville problem

$$r(a) = 0 \ \text{ and } \ A_1\, y(a) + B_1\, y'(a) = 0, r(b) = 0 \ \text{ and } \ A_2\, y(b) + B_2\, y'(b) = 0.$$

- Periodic Sturm-Liouville problem

$$r(a) = r(b) = 0 \ \text{ and } \ \text{no BC}'s, \ \ r(a) = r(b) \ \text{ and } \ y(a) = y(b), y'(a) = y'(b)$$

- For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \boldsymbol{y}$. Consider operator $\boldsymbol{A} \in \mathbb{R}^{m \times m}$

$$(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{A}^T \boldsymbol{y})$$

  If $\boldsymbol{A} = \boldsymbol{A}^T$ (symmetric) then $(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{A}\boldsymbol{y})$, same operator is either slot.

- Consider now $u, v: [a, b] \to \mathbb{R}$, the Sturm-Liouville operator

$$\mathcal{L}u = \left(\frac{\mathrm{d}}{\mathrm{d}x}\left[r(x) \frac{\mathrm{d}}{\mathrm{d}x} \right] + [q(x) + \lambda p(x)] \right) u$$

  satisfies $(\mathcal{L}u, v) = (u, \mathcal{L}v)$ and is said to be *self-adjoint* in the *unweighted scalar* product.

- Any ODE $a(x) y'' + b(x) y' + [c(x) + \lambda d(x)] y = 0$ can be made self-adjoint

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^{\int (b/a)\mathrm{d}x} y'] + \left[\frac{c}{a} e^{\int (b/a)\mathrm{d}x} + \lambda \frac{d}{a} e^{\int (b/a)\mathrm{d}x} \right] y = 0$$

- Bessel equation

$$xy'' + y' + \left(a^2x - \frac{n^2}{x}\right)y = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}(xy') + \left(a^2x - \frac{n^2}{x}\right)y = 0,$$

with solutions $J_n(ax), Y_n(ax)$.

- Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)y'] + n(n+1)y = 0,$$

with solutions $P_n(x)$.

- Similar to Fourier series $f(x)$ can be expressed as a linear combination of other basis sets, e.g., Bessel, Legendre functions

$$f(x) = \sum_{n=1}^{\infty} c_n J(a_n x)$$

$$f(x) = \sum_{n=1}^{\infty} c_n P_n(x)$$