Overview

- Separable partial differential equations (PDEs)
- Classical PDEs and BVPs
  - Heat equation $u_t = k u_{xx}$
  - Wave equation $u_{tt} = a^2 u_{xx}$
  - Laplace equation $u_{xx} + u_{yy} = 0$

- Recall that an $n^{\text{th}}-\text{order}$ ODE is stated as

$$\text{Explicit:} \quad y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \quad \text{Implicit:} \quad F(x, y, y', \ldots, y^{(n)}) = 0$$

- A partial differential equation (PDE) arises for multiple independent variables
- Linear PDE for $u(x, y)$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{1}$$

  with $A, \ldots, G$ functions of x, y

- General solution of (1) is quite difficult, but there exist particular solution procedures of wide utility

- Assume that $u(x, y)$ can be written as a product $u(x, y) = X(x)Y(y)$

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial x \partial y} = X'Y' \quad \frac{\partial^2 u}{\partial y^2} = XY''.$$

- Useful notation

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

- Example: $u_{xx} = 4u_y$. Assuming $u(x, y) = X(x)Y(y)$ leads to $X''Y = 4XY'$

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda,$$

with λ a constant

- $\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda \Rightarrow X'' + 4\lambda X = 0$ and $Y' + \lambda Y = 0$
- Cases:
  - $\lambda = 0$. $X'' = 0 \Rightarrow X(x) = c_1 + c_2 x$, $Y' = 0 \Rightarrow Y(y) = c_3$
  - $\lambda = -\alpha^2 < 0$. $X'' - 4\alpha^2 X = 0 \Rightarrow X(x) = c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$. $Y' - \alpha^2 Y = 0 \Leftrightarrow Y' = \alpha^2 Y \Rightarrow Y(y) = c_6 \exp(\alpha^2 y)$
  - $\lambda = \alpha^2 > 0$. $X'' + 4\alpha^2 X = 0 \Rightarrow X(x) = c_7 \cos(2\alpha x) + c_8 \sin(2\alpha x)$.

    $Y' + \alpha^2 Y = 0 \Leftrightarrow Y' = -\alpha^2 Y \Rightarrow Y(y) = c_9 \exp(-\alpha^2 y)$
- Form $u(x,y) = X(x)Y(y)$ for each case:
  - $\lambda = 0$. $u(x,y) = c_3(c_1 + c_2 x) = b_1 + b_2 x$
  - $\lambda = -\alpha^2 < 0$. $u(x,y) = [b_4 \cosh(2\alpha x) + b_5 \sinh(2\alpha x)] \exp(\alpha^2 y)$
  - $\lambda = \alpha^2 > 0$. $u(x,y) = [b_7 \cos(2\alpha x) + b_8 \sin(2\alpha x)] \exp(-\alpha^2 y)$
- Linear PDE, multiple solutions from $u(x,y) = X(x)Y(y)$, $u = \sum_{k=1}^{\infty} u_k$

- The PDE

$$A\,\frac{\partial^2 u}{\partial x^2} + B\,\frac{\partial^2 u}{\partial x \partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + Fu = G$$

  is:

  - hyperbolic if $B^2 - 4AC > 0$
  - parabolic if $B^2 - 4AC = 0$
  - elliptic if $B^2 - 4AC < 0$

- The above terms arise from classification of quadratic forms

$$[\ \partial_x \ \ \partial_y\ ]\left[\begin{array}{cc} A & B/2 \\ B/2 & C \end{array}\right]\left[\begin{array}{c} \partial_x \\ \partial_y \end{array}\right] u + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + Fu = G$$

$$\det\left(\left[\begin{array}{cc} A & B/2 \\ B/2 & C \end{array}\right] - \lambda\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) =$$

- Heat equation $u_t = k u_{xx}$, parabolic, $s = -\xi^2$
- Wave equation $u_{tt} = a^2 u_{xx}$, hyperbolic, $s^2 = a^2 \xi^2 \Leftrightarrow s^2 - a^2\xi^2 = 0$
- Laplace equation $u_{xx} + u_{yy} = 0$, elliptic, $\xi^2 + \eta^2 = 0$

$$\mathcal{L} u = \int_0^\infty u(t) e^{-st}\, \mathrm{d}t = U(s), \mathcal{L}u_t = sU, \mathcal{L}u_{tt} = s^2 U$$

$$\mathcal{F}_x u = \int_{-\infty}^\infty u(x) e^{-i\xi x}\, \mathrm{d}x = \hat{u}(\xi), \mathcal{F}u_x = -i\xi\,\hat{u}, \mathcal{F}u_{xx} = -\xi^2\,\hat{u}$$

$$\mathcal{F}_y u = \int_{-\infty}^\infty u(x) e^{-i\eta y}\, \mathrm{d}x = \hat{u}(\eta), \mathcal{F}u_y = -i\eta\,\hat{u}, \mathcal{F}u_{yy} = -\eta^2\,\hat{u}$$

- Initial conditions: given at a single point, typically for time variable
  - ODE: $y^{(n)}(t)=f(t,y,y',\dots,y^{(n-1)})$, $y(0)=y_0,\dots,y^{(n-1)}(0)=y_0^{(n-1)}$
  - Time component of PDE: $u_t=u_{xx}$, $u(x,t=0)=u_0(x)$
- Boundary conditions: given at multiple points, typically for space variables
  - ODE (Dirichlet): $y''(x)=f(x,y,y')$, $y(a)=y_a$, $y(b)=y_b$
  - ODE (Neumann): $y''(x)=f(x,y,y')$, $y'(a)=y_a'$, $y'(b)=y_b'$
  - ODE (Robin): $y''(x)=f(x,y,y')$, $A_1 y(a)+B_1\,y'(a)=C$, ...
  - PDE space component (Dirichlet): $u_t=u_{xx}$, $u(a,t)=f(t)$,$u(b,t)=g(t)$
  - PDE space component (Neumann): $u_t=u_{xx}$, $u_x(a,t)=f(t)$, ...
  - PDE space component (Robin): $u_t=u_{xx}$, $A_1 u(a,t)+B_1\,u'(a,t)=f(t)$,...