Overview

- Problem statement
- Separation of variables
- General solution
- Boundary value problems:
  - fixed temperature at ends
  - insulated boundaries

- Rod of length L with initial temperature $f(x)$ and zero temperature at ends
- Denote temperature at time t, position x by $u(x,t)$
- Heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2},\ k \,\text{conductivity}$$

- Initial condition

$$u(x,t=0) = f(x)$$

- Boundary conditions
  - fixed temperature: $u(x=0,t)=u_0$, $u(x=L,t)=u_1$, (e.g., $u_0=u_1=0$)
  - heat flux: $u_x(x=0,t)=q_0$, $u_x(x=L,t)=q_1$, (e.g., $q_0=q_1=0$)

- $u_t = k u_{xx}$, $u(x,0) = f(x)$, $u(0,t) = 0$, $u(L,t) = 0$. (Dirichlet problem)
- $u(x,t) = X(x)\,T(t) \Rightarrow \; XT' = k X'' T$

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

  - $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = c_1 + c_2 x$, $T' = 0 \Rightarrow T(t) = c_3$.

    $u(x,t) = X(x)T(t) = (c_1 + c_2 x)c_3 = b_1 + b_2\, x$

    $u(0,t) = b_1 = 0$, $u(L,t) = b_2\, L = 0 \Rightarrow b_2$. Only solution $u(x,t) = 0$ ✠
  - $\lambda = -\alpha^2 < 0 \Rightarrow X'' - \alpha^2 X = 0 \Rightarrow X(x) = c_4 \cosh(\alpha x) + c_5 \sinh(\alpha x)$

    $T' = k\alpha^2 T \Rightarrow T(t) = c_6 \exp[k\alpha^2\, t]$ (blow-up) ✠
  - $\lambda = \alpha^2 > 0 \Rightarrow X'' + \alpha^2 X = 0 \Rightarrow X(x) = c_7 \cos(\alpha x) + c_8 \sin(\alpha x)$,

    $T' = -k\alpha^2 T \Rightarrow T(t) = c_6 \exp[-k\alpha^2\, t]$

- $u_t = k u_{xx}$, $u(x,0) = f(x)$, $u(0,t) = 0$, $u(L,t) = 0$. (Dirichlet problem)
- $u(x,t) = X(x)\,T(t) \Rightarrow \frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda = -\alpha^2$
- Only possible choice of $\lambda = \alpha^2 > 0 \Rightarrow$

$$u(x,t) = [b_1 \cos(\alpha x) + b_2 \sin(\alpha x)] \exp[-k\alpha^2 t]$$

  - $u(0,t) = 0 \Rightarrow b_1 \exp[-k\alpha^2 t] = 0 \Rightarrow b_1 = 0$
  - $u(L,t) = 0 \Rightarrow b_2 \sin(\alpha L) \exp[-k\alpha^2 t] = 0 \Rightarrow b_2 \sin(\alpha L) = 0$
    - → $b_2 = 0 \Rightarrow u(x,t) = 0$ ✠
    - → $\sin(\alpha L) = 0 \Rightarrow \alpha L = n\pi \Rightarrow \alpha_n = \frac{n\pi}{L}$ all lead to a possible solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\alpha_n x) \exp[-k\alpha_n^2 t]$$

- $u(x, t=0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(\alpha_n x)$,

$$b_n = (f(x), \sin(\alpha_n x)) = \frac{2}{L}\int_0^L f(x), \sin(\alpha_n x)\,\mathrm{d}x$$

- Solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(\alpha_n x) \exp[-k\alpha_n^2 t] \checkmark$$

- $u_t = k u_{xx}$, $u(x,0) = f(x)$, $u_x(0,t) = 0$, $u_x(L,t) = 0$. (Neumann problem)
- Only possible choice of $\lambda = \alpha^2 > 0 \Rightarrow$

$$u(x,t) = [b_1 \cos(\alpha x) + b_2 \sin(\alpha x)] \exp[-k\alpha^2 t]$$

  - $u_x(0,t) = 0 \Rightarrow \alpha[b_2] \exp[-k\alpha^2 t] = 0 \Rightarrow b_2 = 0$
  - $u_x(L,t) = 0 \Rightarrow \alpha[-b_1 \sin(\alpha L)] \exp[-k\alpha^2 t] = 0 \Rightarrow \alpha_n = \frac{n\pi}{L}$

$$u_n(x,t) = \sum_{n=1}^{\infty} b_n \cos(\alpha_n x) \exp[-k\alpha_n^2 t] + b_0$$

$$b_n = (f(x), \cos(\alpha_n x)) = \frac{2}{L} \int_0^L f(x) \cos(\alpha_n x)\, \mathrm{d}x$$

$$b_0 = (f(x), \cos(\alpha_n x)) = \frac{1}{L} \int_0^L f(x)\, \mathrm{d}x$$

- $u_t = k u_{xx}$, $u(x,0) = f(x)$,

  $u(0,t) + a u_x(0,t) = b$, $u(L,t) + c u_x(L,t) = d$. (constant value Robin problem)

  $a, b, c, d \in \mathbb{R}$, always has a solution
- Variable value Robin problem $a(t), b(t), c(t), d(t)$, sometimes has a solution