Overview

- Problem statement
- Separation of variables
- General solution
- Boundary value problems:
  - Mixed Neumann-Dirichlet
  - Dirichlet
  - Inhomogeneous Dirichlet

- Steady-state temperature or static deformation in a plate of size $a \times b$

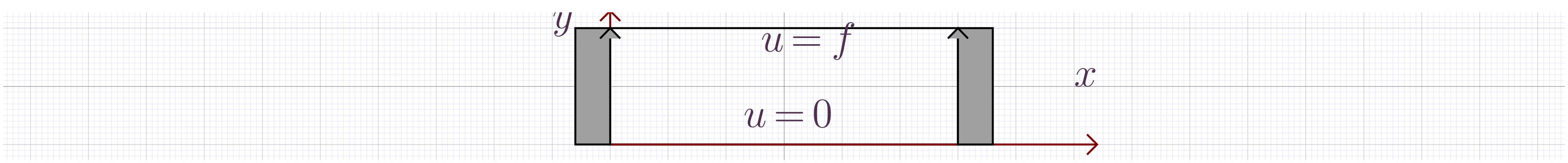
$$u_t = k(u_{xx} + u_{yy}) \quad u_{tt} = a^2(u_{xx} + u_{yy})$$
$$u_{xx} + u_{yy} = 0 \qquad u_{xx} + u_{yy} = 0$$

- Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- Boundary conditions: thermally isolated at $x = 0$, $x = a$. Fixed temperature at $y = 0$, $y = b$.

$$u_x(0, y) = 0, u_x(a, y) = 0, u(x, 0) = 0, u(x, b) = f(x)$$



- $u_{xx} + u_{yy} = 0$, $u(x, 0) = 0$, $u(x, b) = f(x)$, $u_x(0, y) = 0$, $u_x(a, y) = 0$.
- $u(x, y) = X(x)Y(y) \Rightarrow \ X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

- Implications of separation of variables on boundary conditions

$$\begin{aligned} u(x, 0) &= X(x)Y(0) = 0 \Rightarrow Y(0) = 0, \\ u_x(0, y) &= X'(0)Y(y) = 0 \Rightarrow X'(0) = 0, \\ u_x(a, y) &= X'(a)Y(y) = 0 \Rightarrow X'(a) = 0 \end{aligned}$$

- Consider Sturm-Liouville problem $X'' + \lambda X = 0, X'(0) = 0, X'(a) = 0$
  - $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X = c_1 + c_2 x$. $X'(0) = c_2 = 0$, $X'(a) = c_2 = 0$. $X(x) = c_1$. $u(x, y) = c_1 Y(y)$. $Y'' = 0 \Rightarrow Y(y) = c_3 + c_4 y$, $Y(0) = c_3 = 0$. $Y(y) = c_4 y$. $u(x, b) = c_1 Y(b) = c_1 c_4 b = f(x)$

- $u_{xx} + u_{yy} = 0$, $u(x,0) = 0$, $u(x,b) = f(x)$, $u_x(0,y) = 0$, $u_x(a,y) = 0$.
- $u(x,y) = X(x)\,Y(y) \Rightarrow \ X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

- Consider Sturm-Liouville problem $X'' + \lambda X = 0, X'(0) = 0, X'(a) = 0$
  - $\lambda = \alpha^2 > 0 \Rightarrow X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. $X'(0) = \alpha c_2 = 0 \Rightarrow c_2 = 0$. $X'(a) = -\alpha c_1 \sin(\alpha a) = 0 \Rightarrow \alpha_n a = n\pi \Rightarrow \alpha_n = n\pi / a$.

    $Y'' - \alpha^2 Y = 0 \Rightarrow Y(y) = c_3 \cosh(\alpha y) + c_4 \sinh(\alpha y). Y(0) = c_3 = 0$.

    $u(x,b) = X(x)Y(b) = f(x) \Rightarrow \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} b\right) = f(x)$

- $u_{xx} + u_{yy} = 0$, $u(x, 0) = 0$, $u(x, b) = f(x)$, $u_x(0, y) = 0$, $u_x(a, y) = 0$.
- $u(x, y) = X(x)\,Y(y) \Rightarrow \; X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

- Implications of separation of variables on boundary conditions

$$\begin{aligned} u(x, 0) &= X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0, \\ u_x(0, y) &= X'(0)Y(y) = 0 \Rightarrow X'(0) = 0, \\ u_x(a, y) &= X'(a)Y(y) = 0 \Rightarrow X'(a) = 0 \end{aligned}$$

- Consider Sturm-Liouville problem $X'' + \lambda X = 0, X'(0) = 0, X'(a) = 0$
  - $\lambda = -\alpha^2 < 0 \Rightarrow X(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$. $X'(0) = \alpha c_2 = 0 \Rightarrow c_2 = 0$, $X'(a) = \alpha c_1 \sinh(\alpha a) = 0$. No non-trivial solution. ✠

- $u_{xx} + u_{yy} = 0$, $u(x,0) = 0$, $u(x,b) = f(x)$, $u_x(0,y) = 0$, $u_x(a,y) = 0$.

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cos(\alpha_n x) \sinh(\alpha_n y)$$

$$u(x,b) = \sum_{n=1}^{\infty} A_n \cos(\alpha_n x) \sinh(\alpha_n b) = f(x)$$

$$A_n \sinh(\alpha_n h) = (f(x), \cos(\alpha_n x)) = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi}{a} x\right) \mathrm{d}x \Rightarrow$$

$$A_n = \frac{2}{a \sinh(\alpha_n h)} \int_0^a f(x) \cos\left(\frac{n\pi}{a} x\right) \mathrm{d}x\,.$$

- $u_{xx} + u_{yy} = 0$, $u(x,0) = 0$, $u(x,b) = f(x)$, $u(0,y) = 0$, $u(a,y) = 0$.
- $u(x,y) = X(x)Y(y) \Rightarrow \ X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

- Implications of separation of variables on boundary conditions

$$\begin{aligned} u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0, \\ u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0, \\ u(a,y) = X(a)Y(y) = 0 \Rightarrow X(a) = 0 \end{aligned}$$

- Consider Sturm-Liouville problem $X'' + \lambda X = 0, X(0) = 0, X(a) = 0$
  - $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = c_1 + c_2 x$. $X(0) = c_1 = 0$. $X(a) = c_2 a = 0 \Rightarrow c_2 = 0$

- $u_{xx}+u_{yy}=0$, $u(x,0)=0$, $u(x,b)=f(x)$, $u(0,y)=0$, $u(a,y)=0$.
- $u(x,y)=X(x)Y(y)\Rightarrow\ X''Y+XY''=0$

$$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda$$

- Implications of separation of variables on boundary conditions

$$\begin{aligned}u(x,0)=X(x)Y(0)=0\Rightarrow Y(0)=0,\\ u(0,y)=X(0)Y(y)=0\Rightarrow X(0)=0,\\ u(a,y)=X(a)Y(y)=0\Rightarrow X(a)=0\end{aligned}$$

- Consider Sturm-Liouville problem $X''+\lambda X=0, X(0)=0, X(a)=0$
  - $\lambda=-\alpha^2<0\Rightarrow X(x)=c_1\cosh(\alpha x)+c_2\sinh(\alpha x)$. $X(0)=c_1=0$. $X(a)=c_2\sinh(\alpha a)=0\Rightarrow c_2$ ✠

- $u_{xx}+u_{yy}=0$, $u(x,0)=0$, $u(x,b)=f(x)$, $u(0,y)=0$, $u(a,y)=0$.
- Consider Sturm-Liouville problem $X''+\lambda X=0, X(0)=0, X(a)=0$
  - $\lambda=\alpha^2>0 \Rightarrow X(x)=c_1\cos(\alpha x)+c_2\sin(\alpha x)$. $X(0)=c_1=0$. $X(a)=c_2\sin(\alpha a)=0 \Rightarrow \alpha_n a=n\,\pi$.

    $Y''-\alpha^2 Y=0 \Rightarrow Y(y)=c_3\cosh(\alpha y)+c_4\sinh(\alpha y)$.

    $u(x,0)=X(x)Y(0)=X(x)\,c_3=0 \Rightarrow c_3=0$

    $u(x,b)=f(x)=X(x)Y(b)=X(x)\,c_4\sinh(\alpha b)$

$$A_n=\frac{2}{a\sinh(\alpha_n b)}\int_0^a f(x)\sin(\alpha_n x)\,\mathrm{d}x$$

$$u(x,y)=\sum_{n=1}^{\infty} A_n\sin(\alpha_n x)\sinh(\alpha_n y)$$

- $u_{xx} + u_{yy} = 0$, $u(x,0) = f(x)$, $u(x,b) = F(x)$, $u(0,y) = g(y)$, $u(a,y) = G(y)$.
- Superposition of solution to two problems:
  - $u_{xx} + u_{yy} = 0$, $u(x,0) = 0$, $u(x,b) = 0$, $u(0,y) = g(y)$, $u(a,y) = G(y)$ has solution $u_1(x,y)$
  - $u_{xx} + u_{yy} = 0$, $u(x,0) = f(x)$, $u(x,b) = F(x)$, $u(0,y) = 0$, $u(a,y) = 0$ has solution $u_2(x,y)$
- The $u(x,y) = u_1(x,y) + u_2(x,y)$

$$\Delta u = \Delta u_1 + \Delta u_2 = 0$$

$$u(x,0) = u_1(x,0) + u_2(x,0) = u_2(x,0) = f(x)$$