Overview

- Definition
- Substitutions and superposition

- A boundary-value problem for a partial differential equations is *nonhomogeneous* if either the PDE or the boundary conditions are nonhomogeneous:

  $ku_{xx} + F(x) = u_t$ for $0 < x < L$, $t > 0$, $u(0,t) = u_0$, $u(L,t) = u_1$, $t > 0$, $u(x,0) = f(x)$, $0 < x < L$ with $k > 0$

- Separation of variables does not work when applied directly to the full problem. For linear PDEs a solution is obtained by substitution $u(x,t) = v(x,t) + \psi(x)$, and superposition of solution from two problems

  A $k\psi'' + F(x) = 0$, $0 < x < L$, $\psi(0) = u_0$, $\psi(L) = u_1$ (steady-state eq)

  B $kv_{xx} = v_t$, $0 < x < L$, $t > 0$, $v(0,t) = 0$, $v(L,t) = 0$, $v(x,0) = f(x) - \psi(x)$

  (transient eq)

- Constant heat source: $ku_{xx} + r = u_t$ for $0 < x < L$, $t > 0$, $u(0,t) = u_0$, $u(L,t) = u_1$, $t > 0$, $u(x,0) = f(x)$, $0 < x < L$ with $k > 0$

A $k\psi'' + r = 0$, $0 < x < L$, $\psi(0) = u_0$, $\psi(L) = u_1$ (steady-state eq)

$$\psi'' = -\frac{r}{k} \Rightarrow \psi(x) = -\frac{r}{k}x^2 + Ax + u_0, \psi(L) = u_1 = -\frac{r}{k}L^2 + AL + u_0 \Rightarrow$$

$$\frac{u_1 - u_0}{L} = -\frac{r}{k}L + A \Rightarrow A = \frac{u_1 - u_0}{L} + \frac{r}{k}L \Rightarrow$$

$$\psi(x) = -\frac{r}{k}x^2 + \left(\frac{u_1 - u_0}{L} + \frac{r}{k}L\right)x + u_0$$

B $kv_{xx} = v_t$, $0 < x < L$, $t > 0$, $v(0,t) = 0$, $v(L,t) = 0$, $v(x,0) = f(x) - \psi(x)$

- Steady-state solution

$$\psi(x) = -\frac{r}{k}x^2 + \left(\frac{u_1 - u_0}{L} + \frac{r}{k}L\right)x + u_0$$

- Transient problem

$$kv_{xx} = v_t,\ 0 < x < L,\ t > 0,\ v(0,t) = 0,\ v(L,t) = 0,\ v(x,0) = f(x) - \psi(x)$$

$$v(x,t) = X(x)T(t) \Rightarrow k\frac{X''}{X} = \frac{T'}{T} \Rightarrow$$

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(\alpha_n x) \exp[-k\alpha_n^2 t], \alpha_n = \frac{n\pi}{L}$$

$$b_n = (f - \psi, \sin(\alpha_n x)) = \frac{2}{L}\int_0^L [f(x) - \psi(x)]\sin(\alpha_n x)\,\mathrm{d}x$$

Nonhomogeneous PDE BVP example. Time-dependent BCs

- $u_t = u_{xx}$, $0 < x < 1$, $t > 0$, $u(0,t) = \cos t$, $u(1,t) = 0$, $u(x,0) = 0$

$$u(x,t) = X(x)\,T(t) \Rightarrow \frac{T'}{T} = \frac{X''}{X} = -a^2 \Rightarrow$$

$$X(x) = A\cos(ax) + B\sin(ax) \Rightarrow$$

$$X(0) = A = \cos t \maltese \text{ (separation of variables fails)}$$

- Use superposition and substitution

$$u(x,t) = v(x,t) + \psi(x,t)$$

- Problem: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$, $u(0,t) = \cos t$, $u(1,t) = 0$, $u(x,0) = 0$
- Superposition: $u(x,t) = v(x,t) + \psi(x,t)$

$$v_t + \psi_t = v_{xx} + \psi_{xx}$$

$$v(x,0) = -\psi(x,0), \psi(0,t) = u_0(t), \psi(L,t) = u_1(t)$$

- Define problems for ψ, v:

  A $v_t = v_{xx}$

  B $\psi_t = \psi_{xx}$

- Alternative splittings: $v_t = \psi_{xx}$, $\psi_t = v_{xx}$; $v_t = 0$, $\psi_t = v_{xx} + \psi_{xx}$; $v_t = -\psi_t$, $v_{xx} = -\psi_{xx}$; an infinity of possible splittings. Seek a solvable, simple choice

- Problem: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$, $u(0,t) = \cos t$, $u(1,t) = 0$, $u(x,0) = 0$
- Superposition: $u(x,t) = v(x,t) + \psi(x,t)$

$$v_t + \psi_t = v_{xx} + \psi_{xx}$$

$$v(x,0) = -\psi(x,0), \psi(0,t) = u_0(t), \psi(1,t) = u_1(t)$$

- Choose a simple form for $\psi(x,t) = u_0(t) + x\,[u_1(t) - u_0(t)]$, $\psi(0,t) = u_0(t) = \cos t$, $\psi(1, t) = u_1(t) = 0$, $\psi(x,t) = (1-x)\cos t$

$$v_t + (x-1)\sin t = v_{xx}$$

$$v(x,0) = x - 1, v(x,1) = 0$$

$$v(x,t) = X(x)T(t) \Rightarrow \frac{T'}{T} + \frac{(x-1)\sin t}{XT} = \frac{X''}{X}, \maltese.$$

- Another approach. Recall inhomogeneous ODE method. To solve

$$m y'' + k y = \cos(\Omega t)$$

first solve homogeneous equation to obtain $y(t) = A\cos(\omega t) + B\sin(\omega t)$ and then a particular solution of the inhomogeneous equation is sought by superposition of eigenfunctions of $\mathcal{L}y = -(k/m)\,y$ with the forcing term ($\mathcal{L} = \mathrm{d}^2/\mathrm{d}t^2$, e.g., $\cos(\omega t)$, $\sin(\omega t)$, $\omega^2 = k/m$). Obtain solution

$$y(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{\cos(\Omega t)}{m(\omega^2 - \Omega^2)}$$

- This suggests looking for a solution in terms of the eigenfunctions of the homogeneous problem

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin(n\pi x), (x-1)\sin t = \sum_{n=1}^{\infty} g_n(t)\sin(n\pi x)$$

- Obtain

$$g_n(t) = 2\sin t \int_0^1 (x-1)\sin(n\pi x)\,\mathrm{d}x.$$

- Replace in equation

$$\sum_{n=1}^{\infty} \left[\dot{v}_n - (n\pi)^2 v_n\right]\sin(n\pi x) = -\sum_{n=1}^{\infty} g_n(t)\sin(n\pi x).$$

- Recall that $\{\sin(\pi x), \sin(2\pi x), \dots\}$ are linearly independent, hence coefficients can be identified to obtain the ODEs

$$\dot{v}_n - (n\pi)^2 v_n = -g_n(t)$$