Overview

- Orthogonal series

Robin boundary condition problem

- Consider heat conduction PDE $ku_{xx} = u_t$ for $0 < x < 1, t > 0$, with BCs

$$u(0,t) = 0, hu(1,t) + u_x(1,t) = 0, u(x,0) = 1$$

- Separation of variables $u(x,t) = X(x)T(t)$ leads to Sturm-Liouville

$$X'' + \alpha^2 X = 0, X(0) = 0, X'(1) + hX(1) = 0.$$

  Boundary conditions of form $au_x + u = 0$ are known as *Robin* BC's.

- Non-trivial solutions of Sturm-Liouville problem $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$,

$$X(0) = 0 \Rightarrow c_1 = 0, X'(1) + hX(1) = 0 \Rightarrow \alpha \cos \alpha + h \sin \alpha = 0 \Rightarrow \tan \alpha = -\frac{\alpha}{h}$$

- The equation $\tan\alpha = -\alpha/h$ has an infinite number of roots denoted α_n
- Note that in contrast to $\sin\beta L = 0 \Rightarrow \beta_n = n\pi/L$, the solutions α_n are not integer multiples of π/L, hence lead to the formation of a new series of orthogonal functions different from the Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin\alpha_n x$$

- Apply initial condition to obtain $1 = \sum_{n=1}^{\infty} A_n \sin\alpha_n x$ with coefficients

$$A_n = \frac{\int_0^1 \sin\alpha_n x \, \mathrm{d}x}{\int_0^1 \sin^2\alpha_n x \, \mathrm{d}x}$$

- Consider wave equation $a^2 u_{xx} = u_{tt}$ with BCs

$$u(0,t) = 0, u_x(1,t) = 0, u(x,0) = x, u_t(x,0) = 0$$

- Separation of variables leads to $X'' + \alpha^2 X = 0, X(0) = 0, X'(1) = 0$ and $T'' + a^2\alpha^2 T = 0$. Apply BCs to $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

$$X(0) = 0 \Rightarrow c_1 = 0, X'(1) = 0 \Rightarrow \cos \alpha = 0 \Rightarrow \alpha_n = \frac{(2n-1)\pi}{2}$$

Again, this an orthogonal but not Fourier series since α_n is not an integer multiple of a constant.

- Solution

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \right] \sin\left[\left(\frac{2n-1}{2}\right)\pi x\right]$$

- Apply initial conditions $u(x,0) = x$, $u_t(x,0) = f(t)$

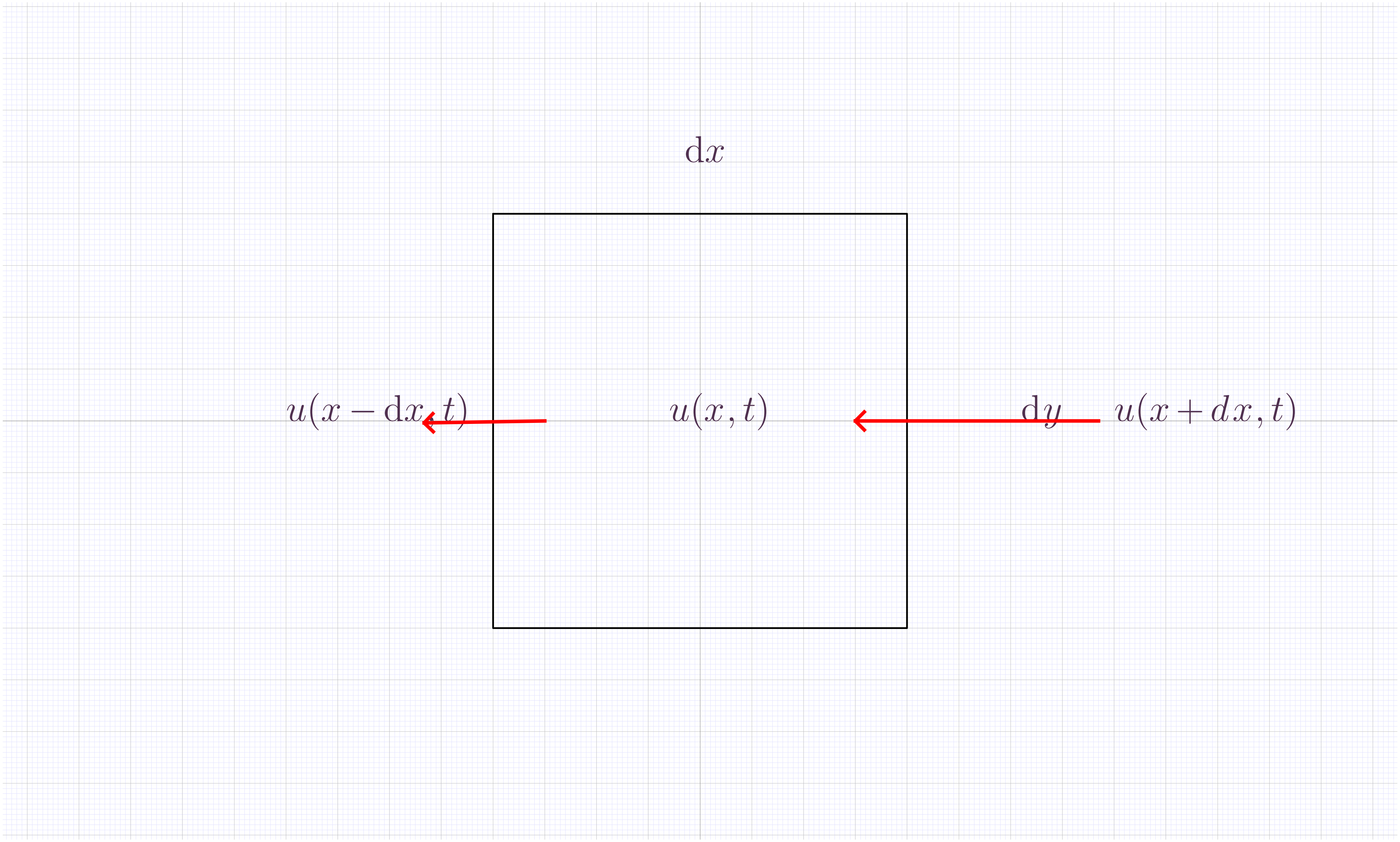
$$u(x,0) = x = \sum_{n=1}^{\infty} A_n \sin\left[\left(\frac{2n-1}{2}\right)\pi x\right]$$

$$A_n = \frac{\int_0^1 x \sin\left[\left(\frac{2n-1}{2}\right)\pi x\right] \mathrm{d}x}{\int_0^1 \sin^2\left[\left(\frac{2n-1}{2}\right)\pi x\right] \mathrm{d}x}$$

- Consider $k u_t = u_{xx} + u_{yy} = \nabla^2 u = \Delta u = \mathrm{div}\,(\mathrm{grad}\, u) = \nabla \cdot \nabla u$
- Separation of variables $u(t, x, y) = T(t)X(x)Y(y)$

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \exp[-k(\alpha_m^2 + \beta_n^2)] \sin(\alpha_m x) \sin(\beta_n y)$$

$$\nabla \cdot \vec{v} = \lim_{|S| \to 0} \frac{\oint_S \vec{v} \cdot \vec{n} \, \mathrm{d}S}{|S|}$$


$\mathrm{d}x$
$u(x-\mathrm{d}x,t)$
$u(x,t)$
$\mathrm{d}y$
$u(x+dx,t)$