Overview

- Polar coordinates
- Cylindrical coordinates

- $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$
- Lamé (metric) coefficients:
  - In Cartesian coordinates $\mathrm{d}x, \mathrm{d}y$ correspond to actual traversed distances

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \begin{bmatrix} \mathrm{d}x & \mathrm{d}y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$$

  - In Polar coordinates traversed distances are $\mathrm{d}r, r\mathrm{d}\theta$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 = \begin{bmatrix} \mathrm{d}r & \mathrm{d}\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \begin{bmatrix} \mathrm{d}r \\ \mathrm{d}\theta \end{bmatrix}$$

- Nabla operator:

$$\text{Cartesian} \quad \nabla = \frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y \quad \text{Polar} \quad \nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{\partial}{r\,\partial\theta}\boldsymbol{e}_\theta$$

Differential operators in polar coordinates

- Gradient of scalar function $f(r,\theta)$

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{\partial f}{r\partial\theta}\boldsymbol{e}_\theta = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{r\partial\theta}\right)$$

- Divergence of vector function $\boldsymbol{V}(r,\theta) = (u(r,\theta), v(r,\theta))$

$$\nabla\cdot\boldsymbol{V}(r,\theta) = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial\theta}(v)\right] = \lim_{|S|\to 0}\frac{\oint_S \boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}S}{|S|}$$

- Laplacian of scalar function $f(r,\theta)$

$$\Delta f = \nabla^2 f = \nabla\cdot\nabla f = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{\partial f}{r\partial\theta}\right)\right] = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

- Dirichlet problem: find $u(r,\theta)$ in a disk of radius a

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \text{ for } r < a,\ u(a,\theta) = f(\theta)$$

- Separation of variables: $u(r,\theta) = R(r)\Theta(\theta) \Rightarrow$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0 \Rightarrow r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) = -\frac{\Theta''}{\Theta} = \alpha^2$$

- Solution is periodic w.r.t θ, $\Theta(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta$ iff $\alpha \in \mathbb{N}$, say $\alpha = n$
- Along r: $r^2 R'' + r R' - n^2 R = 0$ with solution $R(r) = c_3 r^n + c_4 r^{-n}$ for $n \in \mathbb{N}_+$
  - as $r \to 0$, $r^{-n} \to \infty \Rightarrow c_4 = 0$
  - at $r = a$, $R(r) = c r^n$
- For $n = 0$: $r^2 R'' + r R' = 0$ with solution $R(r) = c_5 \cdot 1 + c_6 \ln r \Rightarrow R(r) = c_5$

Series solution for Laplace equation in polar coordinates

- Construct series

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

- Find constants $A_0, A_1, \ldots, B_1, B_2, \ldots$ from Dirichlet B.C. $u(a,\theta) = f(\theta)$

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

- Apply orthogonal Fourier series formulas $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta$

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)\mathrm{d}\theta,\ B_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)\mathrm{d}\theta$$

- Gradient of scalar function $f(r,\theta,z)$

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{\partial f}{r\partial \theta}\boldsymbol{e}_\theta + \frac{\partial f}{\partial z}\boldsymbol{e}_z = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{r\partial \theta}, \frac{\partial f}{\partial z}\right)$$

- Divergence of vector function $\boldsymbol{V}(r,\theta,z) = (u(r,\theta,z), v(r,\theta,z), w(r,\theta,z))$

$$\nabla \cdot \boldsymbol{V}(r,\theta) = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial \theta}(v) + \frac{\partial}{\partial z}(rw)\right] = \lim_{|V|\to 0}\frac{\oint_V \boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}V}{|V|}$$

- Laplacian of scalar function $f(r,\theta)$

$$\Delta f = \nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplace equation in cylindrical coordinates

- Dirichlet problem: find $u(r, \theta, z)$ in a cylinder of radius a, height H

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \text{ for } r < a,$$

$$u(a, \theta, z) = f(\theta, z), u(r, \theta, 0) = g(r, \theta), u(r, \theta, H) = h(r, \theta)$$

- Separation of variables: $u(r, \theta, z) = R(r)\Theta(\theta)Z(z) \Rightarrow$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0$$

- Separation of variables: $u(r,\theta,z) = R(r)\Theta(\theta)Z(z) \Rightarrow$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = -\alpha^2$$

$$Z'' - \alpha^2 Z = 0 \Rightarrow Z(z) = c_1 \cosh \alpha z + c_2 \sinh \alpha z = c_7 e^{\alpha z} + c_8 e^{-\alpha z}$$

$$\alpha = 0 \Rightarrow Z(z) = c_9 + c_{10} z$$

- Remaining equation

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{1}{r^2}\frac{\Theta''}{\Theta} - \alpha^2 \Rightarrow r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \alpha^2\right) = -\frac{\Theta''}{\Theta} = n^2$$

$$\Theta'' + n^2\Theta = 0 \Rightarrow \Theta(\theta) = c_3 \cos n\theta + c_4 \sin n\theta, n \in \mathbb{N}_+$$

$$r^2 R'' + r R' + (\alpha^2 r^2 - n^2)R = 0 \Rightarrow R(r) = c_5 J_n(\alpha r) + c_6 Y_n(\alpha r)$$

- Series solution for finite length cylinder with constant Dirichlet conditions on cylinder faces

$$u(r,\theta,z) = \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) J_n(\alpha r)(1 + C_n z)$$

- Series solution for infinite cylinder

$$u(r,\theta) = \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) J_n(\alpha r)$$

- General series solution for finite length cylinder

$$u(r,\theta,z) = \sum_{n=1}^{\infty} \sum_{\alpha} (A_{n,\alpha} \cos n\theta + B_{n,\alpha} \sin n\theta) J_n(\alpha r)(\cosh \alpha z + C_{n,\alpha} \sinh \alpha z)$$