Overview

- Spherical coordinate Laplace BVP

Spherical coordinates

- $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
- In spherical coordinates traversed distances are $\mathrm{d}r, r\mathrm{d}\theta, r\sin\theta\mathrm{d}\phi$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}^2\phi = \begin{bmatrix} \mathrm{d}r & \mathrm{d}\theta & \mathrm{d}\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix} \begin{bmatrix} \mathrm{d}r \\ \mathrm{d}\theta \\ \mathrm{d}\phi \end{bmatrix}$$

- Nabla operator:

$$\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\boldsymbol{e}_\phi$$

- Gradient of scalar function $f(r,\theta,\phi)$

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{\partial f}{r\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\boldsymbol{e}_\phi = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{r\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)$$

Differential operators in spherical coordinates

- Divergence of vector function $\boldsymbol{V}(r,\theta,\phi) = (u(r,\theta,\phi), v(r,\theta,\phi), w(r,\theta,\phi))$

$$\nabla\cdot\boldsymbol{V}(r,\theta,\phi) = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, u) + \frac{\partial}{\partial\theta}(r\sin\theta\, v) + \frac{\partial}{\partial\phi}(r w)\right]$$

- Laplacian of scalar function $f(r,\theta)$, $\Delta f = \nabla^2 f = \nabla\cdot\nabla f$

$$\Delta f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\,\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}\right)\right]$$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

- Dirichlet problem: find $u(r,\theta)$ in sphere of radius a

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0 \text{ for } r<a,\, u(a,\theta)=f(\theta)$$

- Separation of variables: $u(r,\theta)=R(r)\Theta(\theta)\Rightarrow$

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+\frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta}=0\Rightarrow r^2\left(\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}\right)=-\frac{\Theta''}{\Theta}-\cot\theta\frac{\Theta'}{\Theta}=\lambda$$