Overview

Integral transforms provide solutions to linear PDEs, covering cases in which separation of variables fails

- Error function
- Laplace transform

- Error function, $\mathrm{erf}(x)$, complementary error function $\mathrm{erfc}(x)$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,\mathrm{d}u, \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\,\mathrm{d}u$$

$$\lim_{x \to \infty} \mathrm{erf}(x) = 1, \mathrm{erf}(x) + \mathrm{erfc}(x) = 1$$

- Used for time-unsteady problems, initial value problems
- $f\colon \mathbb{R} \to \mathbb{R}$, the *Laplace transform* is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

and can be considered as the decaying-function analog to a Fourier series

$$f\colon \mathbb{R} \to \mathbb{R},\ f(t+2\pi) = f(t),\ f(t) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} [A_k \cos(kt) + B_k \sin(kt)]$$

- Let $F(s) = \mathcal{L}\{f(t)\}$. Properties

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t = -f(0) + s\,F(s)$$

- Repeated differentiation $F(s) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\{f^{(k)}(t)\} = s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0) - \cdots - f^{(k-1)}(0)$$

- Apply to a PDE, $u_{tt} = a^2 u_{xx}$, $u(x,t)$, $U(x,s) = \int_0^\infty e^{-st} u(x,t)\,\mathrm{d}t$

$$\mathcal{L}\{u_{tt}\} = \mathcal{L}\{a^2 u_{xx}\} \Leftrightarrow \int_0^\infty e^{-st} u_{tt}(x,t)\,\mathrm{d}t = a^2 \int_0^\infty e^{-st} u_{xx}(x,t)\,\mathrm{d}t$$

$$s^2 U(x,s) - s u(x,0) - \frac{\partial u}{\partial t}(x,0) = a^2 \frac{\partial^2}{\partial x^2} U(x,s)$$