Overview

- Wave equation solution
- Heat equation

Apply to a PDE, $u_{tt} = a^2 u_{xx}$ for $0 < x < \pi$, $t > 0$, with initial conditions $u(x, 0) = -\sin 3x$, $u_t(x, 0) = 0$, boundary conditions $u(0, t) = u(\pi, t) = 0$.

– Apply Laplace transform, obtain a family of ODEs labeled by s

$$\begin{aligned} s^2 U(x, s) - s u(x, 0) - \frac{\partial u}{\partial t}(x, 0) &= a^2 \frac{\partial^2}{\partial x^2} U(x, s) \Rightarrow \\ a^2 \frac{\partial^2}{\partial x^2} U(x, s) - s^2 U(x, s) &= s \sin 3x \end{aligned}$$

– Solve the ODE $\Rightarrow U(x, s) = c_1 e^{sx/a} + c_2 e^{-sx/a} - s \sin(3x) / (s^2 + (3a)^2)$

Inverse Laplace transform and apply boundary conditions

$$u(x,t) = \cos(3at)\sin(3x) = \frac{1}{2}[\sin(3(x-at)) + \sin(3(x+at))]$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

- Apply to a PDE, $u_t = a^2 u_{xx}$ for $0 < x < \pi$, $t > 0$, with initial conditions $u(x,0) = -\sin 3x$, boundary conditions $u(0,t) = u(\pi,t) = 0$.
  - Apply Laplace transform, obtain a family of ODEs labeled by s

$$\begin{aligned} sU(x,s) - u(x,0) &= a^2 \frac{\partial^2}{\partial x^2} U(x,s) \Rightarrow \\ a^2 \frac{\partial^2}{\partial x^2} U(x,s) - sU(x,s) &= \sin 3x \end{aligned}$$

  - Solve the ODE $\Rightarrow U(x,s) = c_1 e^{\sqrt{s}\,x/a} + c_2 e^{-\sqrt{s}\,x/a} - \sin(3x)/(s + (3a)^2)$
  - Inverse Laplace transform requires consideration of complex plane extension for general BCs. For homogeneous Dirichlet, $c_1 = c_2 = 0$

$$u(x,t) = \mathcal{L}^{-1}\left[-\frac{\sin(3x)}{s + (3a)^2}\right] = -e^{-9a^2 t} \sin(3x)$$