Overview

- Fourier integral
- Fourier transforms:
  - General Fourier transform
  - Sine and cosine transforms
  - Finite Fourier transform
  - Fast Fourier transform
- Fourier analysis of PDE solutions

- Often used for steady-state problems
- Recall $f(t+T)=f(t)$

$$f(t)=\frac{A_0}{2}+\sum_{k=1}^{\infty}\left[A_k\cos\left(2\pi k\frac{t}{T}\right)+B_k\sin\left(2\pi k\frac{t}{T}\right)\right]$$

- The Fourier coefficients A_k, B_k are obtained as scalar products

$$A_k=\frac{2}{T}\int_0^T f(t)\cos\left(2\pi k\frac{t}{T}\right)\mathrm{d}t,\ B_k=\frac{2}{T}\int_0^T f(t)\sin\left(2\pi k\frac{t}{T}\right)\mathrm{d}t$$

- By analogy define a Fourier transform

$$F(\alpha)=\mathcal{F}\{f(x)\}=\int_{-\infty}^{\infty}f(x)\,e^{i\alpha x}\mathrm{d}x,\ f(x)=\mathcal{F}^{-1}(F(\alpha))=\int_{-\infty}^{\infty}F(\alpha)\,e^{-i\alpha x}\mathrm{d}\alpha$$

- $F(\alpha) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} \mathrm{d}x$, $f(x) = \mathcal{F}^{-1}(F(\alpha)) = \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x} \mathrm{d}\alpha$
- Evaluate $\mathcal{F}\{f'(x)\}$

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)\, e^{i\alpha x} \mathrm{d}x = [f(x)\, e^{i\alpha x}]_{x\to-\infty}^{x\to\infty} - i\,\alpha \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} \mathrm{d}x$$

- Impose condition of f to allow Fourier transform

$$L_2\colon \int_{-\infty}^{\infty} f^2(x)\, \mathrm{d}x \text{ is finite} \Rightarrow [f(x)\, e^{i\alpha x}]_{x\to-\infty}^{x\to\infty} = 0$$

- $\mathcal{F}\{f'(x)\} = -i\,\alpha\, F(\alpha)$, $\mathcal{F}\{f''(x)\} = -\alpha^2\, F(\alpha)$, $\mathcal{F}\{f'''(x)\} = i\,\alpha^3\, F(\alpha)$

- Cosine transform, inverse cosine transform

$$F_c(\alpha) = \mathcal{F}_c\{f\} = \int_0^\infty f(x)\cos\alpha x \,\mathrm{d}x,$$

$$f(x) = \mathcal{F}_c^{-1}\{F_c\} = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\,\mathrm{d}\alpha$$

- Sine transform, inverse sine transform

$$F_s(\alpha) = \mathcal{F}_c\{f\} = \int_0^\infty f(x)\sin\alpha x \,\mathrm{d}x,$$

$$f(x) = \mathcal{F}_s^{-1}\{F_s\} = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\,\mathrm{d}\alpha$$

- $F_c(\alpha) = \mathcal{F}_c\{f(x)\} = \int_0^\infty f(x)\cos\alpha x\,\mathrm{d}x$
- Evaluate $\mathcal{F}_c\{f'(x)\}$

$$\mathcal{F}_c\{f'(x)\} = \int_0^\infty f'(x)\cos\alpha x\mathrm{d}x = [f(x)\sin\alpha x]_{x=0}^{x\to\infty} + \alpha\int_0^\infty f(x)\sin\alpha x\mathrm{d}x$$

$$\mathcal{F}_c\{f'(x)\} = \alpha\mathcal{F}_s\{f(x)\}$$

- Evaluate $\mathcal{F}_c\{f'(x)\}$

$$\mathcal{F}_c\{f''(x)\} = \int_0^\infty f''(x)\cos\alpha x\mathrm{d}x = [f'(x)\sin\alpha x]_{x=0}^{x\to\infty} + \alpha\int_0^\infty f'(x)\sin\alpha x\mathrm{d}x$$

$$\mathcal{F}_c\{f''(x)\} = [f'(x)\cos\alpha x]_{x=0}^{x\to\infty} - \alpha^2\int_0^\infty f(x)\cos\alpha x\mathrm{d}x = -f'(0) - \alpha^2 F_c(\alpha)$$

- $F_s(\alpha) = \mathcal{F}_s\{f(x)\} = \int_0^\infty f(x)\sin\alpha x\,\mathrm{d}x$
- Evaluate $\mathcal{F}_s\{f'(x)\} = \int_0^\infty f'(x)\sin\alpha x\,\mathrm{d}x = \int_0^\infty \sin\alpha x\,\mathrm{d}f(x)$

$$\mathcal{F}_s\{f'(x)\} = [\sin\alpha x\, f(x)]_{x=0}^{x\to\infty} - \alpha\int_0^\infty f(x)\cos\alpha x\,\mathrm{d}x = -\alpha\mathcal{F}_c\{f(x)\}$$

- Evaluate $\mathcal{F}_s\{f''(x)\}$

$$\mathcal{F}_s\{f''\} = -\alpha\mathcal{F}_c\{f'(x)\}$$

$$\mathcal{F}_s\{f''\} = \alpha f(0) - \alpha^2\mathcal{F}_s\{f\}$$

Heat equation

$$k\,\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ -\infty < x < \infty,\ t > 0,\ u(x,0) = f(x) = \begin{cases} u_0 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Fourier transform $\mathcal{F}\{u\} = \int_{-\infty}^{\infty} u(x,t)\, e^{i\alpha x}\,\mathrm{d}x = U(\alpha,t)$,

$$\mathcal{F}\left\{k\,\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} \Rightarrow \mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial U(\alpha,t)}{\partial t} = -k\alpha^2 U \Rightarrow$$

$$U(\alpha,t) = e^{-k\alpha^2 t} U(\alpha,0) = 2\frac{u_0}{\alpha}\sin\alpha\, e^{-k\alpha^2 t} = 2u_0 e^{-k\alpha^2 t}\,\frac{\sin\alpha}{\alpha}$$

$$U(\alpha,0) = \int_{-\infty}^{\infty} u(x,0)\, e^{i\alpha x}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\,\mathrm{d}x = \mathcal{F}\{f\} = u_0 \int_{-1}^{1} e^{i\alpha x}\,\mathrm{d}x$$

$$U(\alpha,0) = \frac{u_0}{i\alpha}\left[e^{i\alpha x}\right]_{x=-1}^{x=1} = \frac{u_0}{i\alpha}\left(e^{i\alpha} - e^{-i\alpha}\right) = 2\,\frac{u_0}{\alpha}\sin\alpha$$

Inverse Fourier transform

$$u(x,t) = \mathcal{F}^{-1}\{U(\alpha,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\alpha,t)\, e^{-i\alpha x}\, \mathrm{d}\alpha \Rightarrow$$

$$u(x,t) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} e^{-k\alpha^2 t}\,\frac{\sin\alpha}{\alpha}\, e^{-i\alpha x}\, \mathrm{d}\alpha = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\, e^{-i\alpha x - k\alpha^2 t}\, \mathrm{d}\alpha$$

$$u(x,t) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\, e^{-\left[\sqrt{kt}\,\alpha + \frac{ix}{2\sqrt{kt}}\right]^2 - \frac{x^2}{4kt}}\, \mathrm{d}\alpha$$