Overview

- Finite Fourier transform
- Fast Fourier transform

- Finite Fourier transforms arise for functions defined on a finite domain

Infinite domain

$$f\colon \mathbb{R} \to \mathbb{R}$$

$$F_c(\alpha) = \mathcal{F}_c\{f\} = \int_0^\infty f(x)\cos\alpha x \,\mathrm{d}x,$$

$$f(x) = \mathcal{F}_c^{-1}\{F_c\} = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\,\mathrm{d}\alpha$$

$$F_s(\alpha) = \mathcal{F}_c\{f\} = \int_0^\infty f(x)\sin\alpha x \,\mathrm{d}x,$$

$$f(x) = \mathcal{F}_s^{-1}\{F_s\} = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\,\mathrm{d}\alpha$$

Finite domain

$$f\colon [0,p] \to \mathbb{R}$$

$$F_c(n) = \int_0^p f(x)\cos\frac{n\pi}{p}x\,\mathrm{d}x$$

$$f(x) = \mathcal{F}_c^{-1}\{F_c\} =$$

$$\frac{2}{p}\left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n)\cos\frac{n\pi}{p}x\right]$$

$$F_s(n) = \int_0^p f(x)\sin\frac{n\pi}{p}x\,\mathrm{d}x$$

$$f(x) = \mathcal{F}_s^{-1}\{F_s\} =$$

$$\frac{2}{p}\sum_{n=1}^{\infty} F(n)\sin\frac{n\pi}{p}x$$

- Recall $f\colon \mathbb{R} \to \mathbb{R}$, f periodic $f(t+T) = f(t)$

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos 2\pi n \frac{t}{T} + B_n \sin 2\pi n\, \frac{t}{T}$$

$$A_n = \left(f, \cos 2\pi n \frac{t}{T}\right), B_n = \left(f, \sin 2\pi n \frac{t}{T}\right)$$

- f was defined with a finite number of points of discontinuity $\forall t$

$$\lim_{N\to\infty} T_n(t) = \lim_{N\to\infty} \frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos 2\pi n \frac{t}{T} + B_n \sin 2\pi n\, \frac{t}{T} = f(t)$$

- What happens at a discontinuity, t_*?

$$T_n(t_*) = \frac{1}{2}(f(t_*^-) + f(t_*^+))$$

- Having defined

$$F_c(n) = \int_0^p f(x) \cos \frac{n\pi}{p} x \, \mathrm{d}x$$

- Compute transform of second derivative

$$\mathcal{F}_{\mathrm{cn}}(f'') = f'(p)(-1)^n - f'(0) - \left(\frac{n\pi}{p}\right)^2 F_c(n)$$

- Compare to inifinite-domain Fourier transform

$$F(\alpha) = \mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} \mathrm{d}x, \mathcal{F}\{f''\} = -\alpha^2 F(\alpha)$$