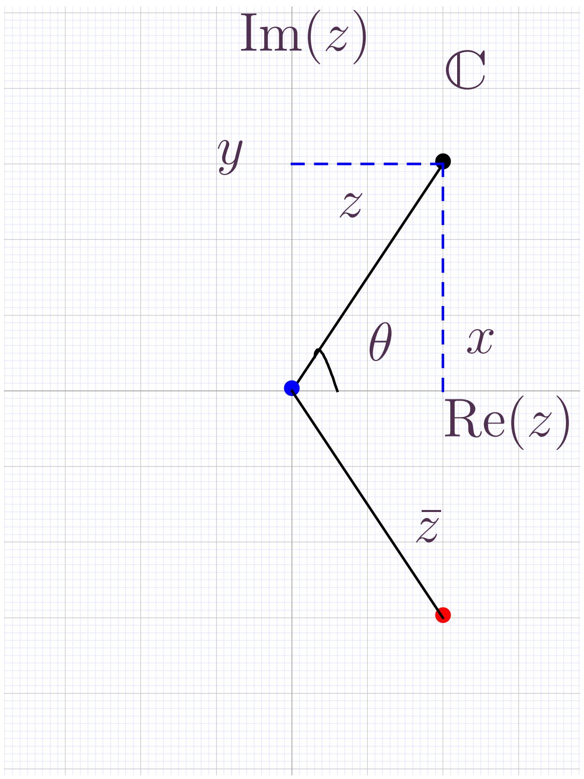
Overview

- Complex numbers
- Powers and roots
- Sets in the complex plane
- Functions of a complex variable

- Cartesian form $z \in \mathbb{C}$, $z = x + i\,y$, $x, y \in \mathbb{R}$
- Operations, $z_1 = x_1 + i\,y_1$, $z_2 = x_2 + i\,y_2$
  - $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
  - $z_1 z_2 = (x_1 y_1 - x_2 y_2) + i(x_1 y_2 + x_2 y_1)$
  - $\bar{z} = x - i\,y$
- Polar form $z = r\,e^{i\theta} = (r\cos\theta) + i\,(r\sin\theta)$
  - $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
  - $\bar{z} = r\,e^{-i\theta}$



- Modulus and argument: $z = r\,e^{i\theta}$, $\mathrm{mod}\,(z) = |z| = r$, $\arg(z) = \theta$
- z^n, $n \in \mathbb{Z}$, $z^n = (r\,e^{i\theta})^n = r^n\,e^{in\theta}$
- Roots, w is n^{th} root of z, $w = \sqrt[n]{z} = z^{1/n}$ if $w^n = z$

$$w = \rho e^{i\varphi}, z = r\,e^{i\theta} \Rightarrow \rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta) \Rightarrow \rho = r^{1/n}$$

$$\cos n\varphi = \cos\theta, \sin n\varphi = \sin\theta \Rightarrow \varphi_k = \frac{\theta + 2k\pi}{n}, k = 0, 1, \ldots, n-1$$

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], k = 0, 1, \ldots, n-1$$

- Example $w = \sqrt[6]{1} \Rightarrow w_k = \cos\left(\frac{\theta + 2k\pi}{6}\right) + i\sin\left(\frac{\theta + 2k\pi}{6}\right)$, $k = 0, 1, \ldots, 5$

- $|z - z_0| = R$, circle of radius R centered at z_0
- $\mathrm{Im}(z) < 0$, lower half plane
- $-1 < \mathrm{Re}(z) < 1$, centered infinite vertical strip of width 2
- $1 < |z| < 2$, circular ring

A *neighborhood* of z_0, $|z - z_0| < \varepsilon$, $\varepsilon \in \mathbb{R}_+$

- z_0 is an *interior point* of set S, if all points of a neighborhood of z_0 are inside S

$$\exists \varepsilon > 0, \forall z \in N(z_0, \varepsilon) = \{z: |z - z_0| < \varepsilon\}, z \in S \Rightarrow z_0 \text{ interior point}$$

- S is *open* if it only contains interior points
- z_0 is an *exterior point* of set S, if all points of a neighborhood of z are outside S

$$\exists \varepsilon > 0, \forall z \in N(z_0, \varepsilon) = \{z: |z - z_0| < \varepsilon\}, z \notin S \Rightarrow z_0 \text{ exterior point}$$

- z_0 is a *boundary point* of set S, every neighborhood of $z \in S$ contains a point in S and a point outside S

$$\forall \varepsilon > 0, \exists z_1 \in N(z_0, \varepsilon), \exists z_2 \in N(z_0, \varepsilon) = \{z: |z - z_0| < \varepsilon\}, z_1 \in S, z_2 \notin S \Rightarrow z_0 \text{ boundary point}$$

Types of sets

- S open is connected if all points can be connected by a polygonal line completely within S
- S is a *domain* if it is open and connected

- $w = f(z) = u(x,y) + iv(x,y)$, $f: \mathbb{C} \to \mathbb{C}$
- Expect some differences w.r.t. real functions, e.g.
  - $g: \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$, $g(x) = \sqrt{x}$
  - $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \sqrt{z}$, careful about possibility of multiple outputs
- Examples:
  - polynomials, e.g., $f(z) = z^4 - 3z$
  - rational functions, e.g. $f(z) = (z^3 - 2)/(z^3 + 2)$
  - $f(z) = z + \mathrm{Re}(z)$