Overview

- Cauchy-Riemann equations
- Exponential and logarithmic functions
- Trigonometric & hyperbolic functions

- $w\colon\mathbb{C}\to\mathbb{C}$, $w(z)=u+iv$, $z=x+iy$, w is differentiable at z_0 iff

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Proof: w differentiable implies existence of limit $\lim_{z\to z_0}\frac{w(z)-w(z_0)}{z-z_0}=w'(z_0)$, irrespective of path $z\to z_0$. Choose two paths $x\to x_0$, $y=y_0$, $x=x_0$, $y\to y_0$

$$\lim_{x\to x_0, y=y_0}\frac{u(x,y_0)-u(x_0,y_0)+i[v(x,y_0)-v(x_0,y_0)]}{x-x_0}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\lim_{x=x_0, y\to y_0}\frac{u(x_0,y)-u(x_0,y_0)+i[v(x_0,y)-v(x_0,y_0)]}{i(y-y_0)}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Identify real and imaginary parts to obtain $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

- Defined as inverse of exponential $w = \ln z$, if $z = e^w$

$$e^w = e^{\ln|z| + i(\theta + 2k\pi)} = e^{\ln|z| + i\theta} = e^{\ln|z|}\, e^{i\theta} = e^{\ln|z|}(\cos\theta + i\sin\theta)$$

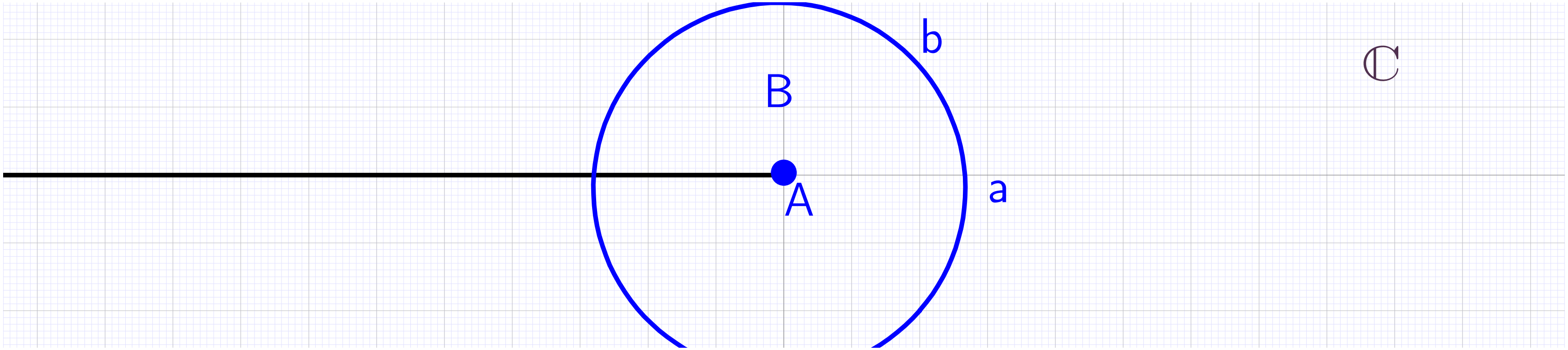
- However: e^w periodic, $e^{w + k2\pi i} = e^w$ implies multiple possible values for $\ln z$

$$\ln z = \ln|z| + i(\theta + 2k\pi)\ (\ln|z| = \log_e|z|)$$

- *Principal value* $\operatorname{Ln} z$ in terms of $\operatorname{Arg} z \in (-\pi, \pi]$

$$k \in \mathbb{Z}, \operatorname{Ln} z = \log|z| + i \operatorname{Arg} z$$

- $\operatorname{Ln} z$ analytic in $D = \mathbb{C} \setminus (-\infty, 0]$, i.e., the complex plane with a branch cut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- $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

  $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $\ln(z_1 z_2) = \ln\left(r_1 r_2 e^{i(\theta_1+\theta_2)}\right) = \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2k\,\pi) = \ln r_1 + \ln r_2 + i(\theta_1 + 2l\pi) + i(\theta_2 + 2m\pi) = \ln z_1 + \ln z_2, l + m = k$

- $\ln(z_1 / z_2) = \ln z_1 - \ln z_2$
- $\ln z^\alpha = \alpha \ln z$

  $z = r\,e^{i\theta}$, $\ln z^\alpha = \ln(r\,e^{i\theta})^\alpha = \ln r^\alpha\, e^{i\alpha\theta} = \ln r^\alpha + i(\alpha\theta + 2k\pi) = \alpha[\ln r + i(\theta + 2k\pi)] = \alpha \ln z$

- $z^\alpha = e^{\alpha \ln z}$

- Defined through exponential function

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz), \sin z = \frac{e^{iz} - e^{-iz}}{2i} = -i\sinh(iz)$$

$$iz = e^{i\pi/2} r e^{i\theta} = r e^{i(\theta+\pi/2)}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} = \cos(iz), \sinh z = \frac{e^{z} - e^{-z}}{2} = -i\sin(iz)$$

- $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions (analytic over $\mathbb{C}$)
- Many identities, most remarkable

$$\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\cos x\sinh y$$

$$\cos z = \cos(x+iy) = \cos x\cos(iy) - \sin(x)\sin(iy) = \cos x\cosh y - i\sin x\sinh y$$

- $w = \sin^{-1} z = -i \ln[iz + (1 - z^2)^{1/2}]$ if $z = \sin w$

$$z = \frac{e^{iw} - e^{-iw}}{2i}, t = e^{iw}, z = \frac{t^2 - 1}{2it} \Rightarrow t^2 - 2izt - 1 = 0$$

$$t_{1,2} = iz \pm \sqrt{-z^2 + 1}, t = iz + \sqrt{1 - z^2} = e^{iw} \Rightarrow iw = \ln\left[iz + \sqrt{1 - z^2}\right] \Rightarrow$$

$$w = -i \ln\left[iz + \sqrt{1 - z^2}\right]$$

- $w = \cos^{-1} z = -i \ln[z + (1 - z^2)^{1/2}]$ if $z = \cos w$
- $w = \sinh^{-1} z = \ln[z + (z^2 + 1)^{1/2}]$ if $z = \sinh w$
- $w = \cosh^{-1} z = \ln[z + (z^2 - 1)^{1/2}]$ if $z = \cosh w$