Overview

- Review of Riemann integral
- Contour integrals in $\mathbb{C}$
- Solved examples

- Consider function $f\colon [a,b] \to \mathbb{R}$
- *Partition* on interval $[a,b]$ is $\mathcal{P} = \{x_i | i = 0, \ldots, n\}$, $a = x_0 < x_1 < \cdots < x_n = b$
- *Norm of partition* $\|\mathcal{P}\| = \max_{0 \leqslant i \leqslant n-1} |x_{i+1} - x_i|$
- *Tagged partition* = partition and set $\mathcal{T} = \{t_i | i = 1, \ldots, n\}$, $t_i \in [x_{i-1}, x_i]$
- *Riemann sum* $S_{\mathcal{P},\mathcal{T}} = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$
- If $\lim_{\|\mathcal{P}\| \to 0} S_{\mathcal{P},\mathcal{T}}$ exists, f is said to be Riemann-integrable and

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

- Consider a curve C in $\mathbb{R}^2$, i.e., a pair of univariate functions $x, y\colon [a,b] \to \mathbb{R}$

$$C\colon x(s), y(s),$$

  s is the *curvilinear parameter*.

- Partition the interval by $\mathcal{P} = \{s_i | i = 0, \ldots, n\}$, $a = s_0 < s_1 < \cdots < s_n = b$, with $\|\mathcal{P}\| = \max_{0 \leqslant i \leqslant n-1} |s_{i+1} - s_i|$
- Form tagged partition from $\mathcal{P}$ and $\mathcal{T} = \{t_i | i = 1, \ldots, n\}$, $t_i \in [s_{i-1}, s_i]$
- Consider $F\colon \mathbb{R}^2 \to \mathbb{R}$ a real, continuous, bivariate function
- Form the sum $S_{\mathcal{P},\mathcal{T}} = \sum_{i=1}^{n} F(x(t_i), y(t_i))(s_i - s_{i-1})$
- If $\lim_{\|\mathcal{P}\| \to 0} S_{\mathcal{P},\mathcal{T}}$ exists, F is said to be *path-integrable*, and

$$\int_C F(x,y)\,\mathrm{d}s = \int_a^b F(x(s), y(s))\,\mathrm{d}s = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{n} F(x(t_i), y(t_i))(s_i - s_{i-1})$$

- An integral on the curve $C: x(s), y(s), x, y: [a, b] \to \mathbb{R}$, can also be stated by explicit dependence on $(x, y) \in \mathbb{R}^2$

$$I = \int_C P(x, y)\,\mathrm{d}x + \int_C Q(x, y)\,\mathrm{d}y = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y$$

- The curve C is smooth if $x, y \in \mathcal{C}^1(a, b)$ in which case $\mathrm{d}x = x'\,\mathrm{d}s, \mathrm{d}y = y'\,\mathrm{d}s$

$$I = \int_a^b [P(x(s), y(s))\,x'(s) + Q(x(s), y(s))y'(s)]\mathrm{d}s$$

- Recall that the vector with components $(x'(s), y'(s))$ is tangent to C

- Analogous to path integral in $\mathbb{R}^2$, let $z(s) = x(s) + iy(s)$, $z\colon [a,b] \to \mathbb{C}$ define a path or *contour* in $\mathbb{C}$.
- Partition C by $\mathcal{P} = \{z_0 = z(s_0), z_1 = z(s_1), \ldots, z_n = z(s_n)\}$, $a = s_0 < \ldots < s_n = b$, with $\|\mathcal{P}\| = \max_{1 \leqslant k \leqslant n} |z_k - z_{k-1}|$, with $\zeta_k = z(t_k)$, $t_k \in [s_{k-1}, s_k]$.
- The path integral of $f\colon \mathbb{C} \to \mathbb{C}$ is

$$\int_C f(z)\,\mathrm{d}z = \lim_{\|\mathcal{P}\| \to 0} \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$$

- With $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$

$$\int_C f(z)\,\mathrm{d}z = \int_C u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_C v\,\mathrm{d}x + u\,\mathrm{d}y = \int_a^b f(z(s))\,z'(s)\mathrm{d}s$$

1. $I = \int_C \bar{z}\,\mathrm{d}z$, $C\colon x(t) = 3t,\ y(t) = t^2,\ 0 \leqslant t \leqslant 2\pi,\ f(z) = \bar{z} = x - i\,y = u + i\,v$

$$C - R\colon u_x = 1 \neq -1 = v_y$$

$$I = \int_C \bar{z}\,z'(t)\mathrm{d}t = \int_0^{2\pi} (x(t) - i\,y(t))(x'(t) + i\,y'(t))\,\mathrm{d}t \Rightarrow$$

$$I = \int_0^{2\pi} (x(t)x'(t) + y(t)y'(t) + i[x(t)y'(t) - x'(t)y(t)])\,\mathrm{d}t$$

$$I = \int_0^{2\pi} [9t + 2t^2]\mathrm{d}t + i\int_0^{2\pi} [6t^2 - 3t^2]\mathrm{d}t = \frac{9}{2}4\pi^2 + \frac{2}{3}8\pi^3 + i\,8\pi^4$$

1. $I = \oint_C \frac{1}{z}\,\mathrm{d}z$, $C\colon x(t) = \cos(t),\, y(t) = \sin(t)$, $0 \leqslant t \leqslant 2\pi$

$$z(t) = e^{it},\, \frac{1}{z} = e^{-it},\, z'(t) = i\,e^{it}$$

$$I = \int_0^{2\pi} e^{-it}\, i\, e^{it}\,\mathrm{d}t = 2\pi i$$

2. $I = \oint_C z\,\mathrm{d}z = 0$, $C\colon x(t) = \cos(t),\, y(t) = \sin(t)$, $0 \leqslant t \leqslant 2\pi$

$$z(t) = e^{it},\, z'(t) = i\,e^{it}$$

$$I = \int_0^{2\pi} e^{it}\, i\, e^{it}\,\mathrm{d}t = i\int_0^{2\pi} e^{2it}\,\mathrm{d}t = \frac{i}{2i}\left[e^{2it}\right]_{t=0}^{t=2\pi} = 0$$

- $\int_C a f(z)\,\mathrm{d}z = a\int_C f(z)\,\mathrm{d}z$, $a \in \mathbb{C}$ a constant
- $\int_C [\,f(z) + g(z)]\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z + \int_C g(z)\,\mathrm{d}z$
- $\int_C f(z)\,\mathrm{d}z = -\int_{-C} f(z)\,\mathrm{d}z$, $-C$ denotes contour in reverse direction
- If f continuous on smooth curve C and $|\,f(z)| \leqslant M$ for $\forall z \in C$ then

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leqslant ML$$

with L the curve length

- Example

$$I = \oint_{|z|=4} \frac{e^z}{z+1} \mathrm{d}z$$

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

$$|z_1| \leqslant |z_1 + z_2 + (-z_2)| \leqslant |z_1 + z_2| + |z_2| \Rightarrow |z_1| - |z_2| \leqslant |z_1 + z_2|$$

$$\left|\frac{e^z}{z+1}\right| = \frac{|e^z|}{|z+1|} \leqslant \frac{|e^z|}{|z|-1} = \frac{|e^x e^{iy}|}{|z|-1} = \frac{|e^x|}{|z|-1} \leqslant \frac{e^4}{3} \Rightarrow I \leqslant \frac{e^4}{3} 2\pi 4 = \frac{8\pi e^4}{3}$$