Overview

- Bounding integral values
- Cauchy-Goursat theorem
- Cauchy's integral formulas

- If f continuous on smooth curve C and $|f(z)| \leqslant M$ for $\forall z \in C$ then

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leqslant ML$$

with L the curve length

- Example

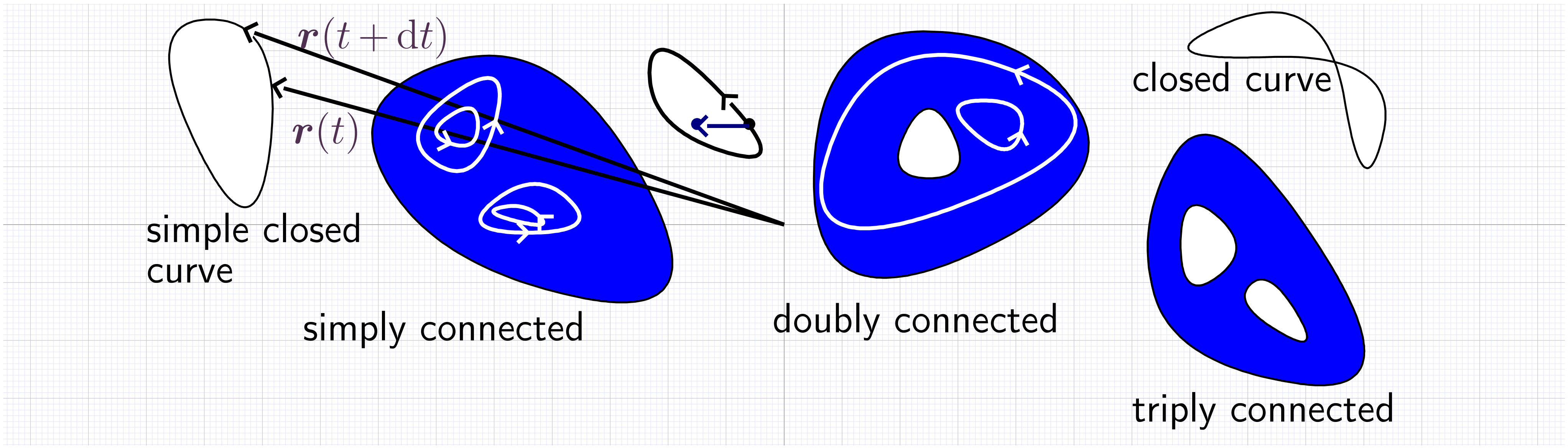
$$I = \oint_{|z|=4} \frac{e^z}{z+1}\,\mathrm{d}z$$

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

$$|z_1| \leqslant |z_1 + z_2 + (-z_2)| \leqslant |z_1 + z_2| + |z_2| \Rightarrow |z_1| - |z_2| \leqslant |z_1 + z_2|$$

$$\left|\frac{e^z}{z+1}\right| = \frac{|e^z|}{|z+1|} \leqslant \frac{|e^z|}{|z|-1} = \frac{|e^x e^{iy}|}{|z|-1} = \frac{|e^x|}{|z|-1} \leqslant \frac{e^4}{3} \Rightarrow I \leqslant \frac{e^4}{3} 2\pi 4 = \frac{8\pi e^4}{3}$$

- Consider *contour integrals*, i.e., integrals over a simple closed curve C $I = \oint_C f(z)\,\mathrm{d}z$, $\boldsymbol{r}(t) = x(t)\boldsymbol{e}_x + y(t)\boldsymbol{e}_y$, $\boldsymbol{\tau} = \boldsymbol{r}'(t)$, $f\colon [a,b] \to \mathbb{R}$, $f'\colon (a,b) \to \mathbb{R}$
- Recall: a *domain* in $\mathbb{C}$ is an open and connected set within $\mathbb{C}$
  - a domain is *simply connected* if any contour can be shrunk to a point without leaving the domain (domain has no "holes")
  - otherwise the domain is *multiply connected*, e.g., doubly connected, triply connected, etc.



- Closed curve $C: \boldsymbol{r}(s)$, tangent, normal vectors $\boldsymbol{t}(s), \boldsymbol{n}(s)$, $f: \mathbb{C} \to \mathbb{C}$, $f = u + iv$, $\boldsymbol{v} = u\boldsymbol{e}_x + v\boldsymbol{e}_y$, $\boldsymbol{v}: \mathbb{R}^2 \to \mathbb{R}^2$
  - $\Gamma = \oint_c \boldsymbol{v}(x(s), y(s)) \cdot \boldsymbol{t}(s)\, \mathrm{d}s$ is the *circulation* of $\boldsymbol{v}$ on curve C.

$$\oint_c \boldsymbol{v}(x(s), y(s))\, \boldsymbol{t}(s)\, \mathrm{d}s = \oint_c u\, \mathrm{d}x + v\, \mathrm{d}y$$

  - $\Phi = \oint_c \boldsymbol{v}(x(s), y(s)) \cdot \boldsymbol{n}(s)\, \mathrm{d}s$ is the *flux* of $\boldsymbol{v}$ across curve C.

$$\oint_c \boldsymbol{v}(x(s), y(s))\, \boldsymbol{n}(s)\, \mathrm{d}s = \oint_c u\, \mathrm{d}y - v\, \mathrm{d}x$$

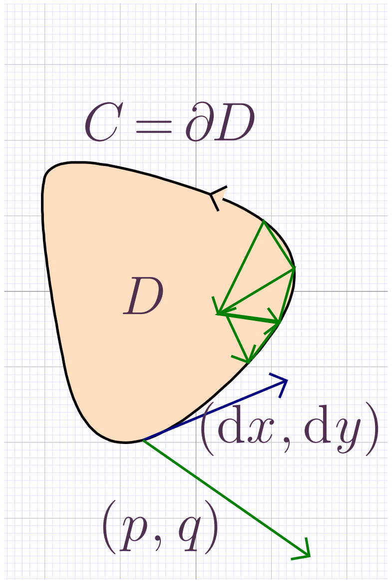
  - In $\mathbb{C}$: $\Psi = \Gamma + i\Phi = \oint_c (u - iv)(\mathrm{d}x + i\,\mathrm{d}y) = \oint_c \overline{f(z)}\, \mathrm{d}z$, $\Gamma = \mathrm{Re}(\Psi)$, $\Phi = \mathrm{Im}(\Psi)$
- Green's theorem

$$\oint_{C = \partial D} p\, \mathrm{d}x + q\, \mathrm{d}y = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) \mathrm{d}A$$

Theorem. *If* f *is analytic in a simply connected domain* $D \subseteq \mathbb{C}$ *and* f' *is continuous then* $I = \oint_C f(z)\,\mathrm{d}z = 0$ *for any contour* C *within* D.

$$\oint_C f(z)\,\mathrm{d}z = \oint_C [u + iv]\mathrm{d}(x + iy) \Rightarrow$$

$$\oint_C f(z)\,\mathrm{d}z = \oint_C [u\,\mathrm{d}x - v\,\mathrm{d}y] + i\oint_C [v\,\mathrm{d}x + u\,\mathrm{d}y].$$



Green's theorem: for p, q and first derivatives p_y, q_x continuous

$$\oint_{C=\partial D} p\,\mathrm{d}x + q\,\mathrm{d}y = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\mathrm{d}A$$

But Cauchy-Riemann $\Rightarrow u_x = v_y, u_y = -v_x$.

- Continuity of f' not required

Theorem. *If f is analytic in a simply connected domain $D \subseteq \mathbb{C}$ then $I = \oint_C f(z)\,\mathrm{d}z = 0$ for any contour C within D.*

- $\oint_C e^z\,\mathrm{d}z = 0$
- $\oint_C z^n\,\mathrm{d}z = 0, n \in \mathbb{N}$
- $\oint_{|z-2|=1} z^{-2}\,\mathrm{d}z = 0$
- $\oint_{|z-2|=1} z^{-1}\,\mathrm{d}z = 0$
- $\oint_{|z|=1} z^{-1}\,\mathrm{d}z = 2\pi i \neq 0$

- $\oint_{|z-i|=1} \frac{1}{z-i}\,\mathrm{d}z = 2\pi i$
- $\oint_{|z-2|=2} \frac{5z+7}{z^2+2z-3}\,\mathrm{d}z = 6\pi i$
- $\oint_{|z|=3} \frac{1}{z^2+1}\,\mathrm{d}z = 0$

- f analytic in domain D, and C a simple closed contour in D

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \tag{1}$$

- Proof: Let C_ε be circle of radius ε around z_0

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_{C_\varepsilon} \frac{f(z) - f(z_0) + f(z_0)}{z - z_0}\, dz = M\varepsilon + f(z_0) \oint_{C_\varepsilon} \frac{1}{z - z_0}\, dz \Rightarrow$$

$$\oint_C \frac{f(z)}{z - z_0}\, dz = M\varepsilon + f(z_0)\, 2\pi i, \text{take limit } \varepsilon \to 0 \Rightarrow (1)$$

- f analytic along with derivatives up to order n in domain D, and C a simple closed contour in D

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \tag{2}$$

- Examples:
  - $\oint_C \frac{z+1}{z^4+4z^3} \, \mathrm{d}z = -\frac{3\pi}{32} i$, $C\colon |z| = 1$
  - $\oint_C \frac{z^3+3}{z(z-i)^2} \, \mathrm{d}z = -6\pi i$, C: figure “8” enclosing $z_1 = 0$, $z_2 = i$