Overview

- Sequences and series in $\mathbb{C}$
- Taylor series

- Sequence $\{z_n\}$ converges to $L \in \mathbb{C}$ iff $\operatorname{Re} z_n \to \operatorname{Re} L$ and $\operatorname{Im} z_n \to \operatorname{Im} L$
- $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, s.t. n > n_\varepsilon, |z_n - L| < \varepsilon$, $z_n = x_n + i\,y_n$, $L = M + i\,N$

$$|z_n - L| = [(x_n - M)^2 + (y_n - N)^2]^{1/2}$$

- Examples:
  - $z_n = 5\,i^n$, Convergent? No
  - $z_n = 1 + e^{n\pi i}$, Convergent? No
  - $z_n = \frac{n + i^n}{\sqrt{n}}$, Convergent? No
  - $z_n = \frac{i^n + 1}{n + i}$, Convergent? Yes
- Motivation for sequences: ensure closure of $\mathbb{R}$, $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$, $x_n = \left(1 + \frac{1}{n}\right)^n \to e$

- The series $\sum_{k=1}^{\infty} z_k$ is *convergent* if the sequence of partial sums S_n converges

$$S_n = \sum_{k=1}^{n} z_k, \{S_n\}_{n \in \mathbb{N}}$$

- The geometric series $\sum_{k=1}^{\infty} z^{k-1}$ converges to $1/(1-z)$ when $|z| < 1$

$$S_n = \sum_{k=1}^{n} z^{k-1} = 1 + z + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z} \to \frac{1}{1 - z}$$

- Note that

$$\frac{1}{1 - z} = 1 + z + \cdots + z^{n-1} + \frac{z^n}{1 - z}$$

of interest in applications of Cauchy's integral formula $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}z$

- If $\sum_{k=1}^{\infty} z_k$ converges then $\lim_{n\to\infty} z_n = 0$
- If $\lim_{n\to\infty} z_n \neq 0$ then $\sum_{k=1}^{\infty} z_k$ diverges
- $\sum_{k=1}^{\infty} z_k$ is *absolutely convergent* if $\sum_{k=1}^{\infty} |z_k|$ is convergent
- Ratio test: series $\sum_{k=1}^{\infty} z_k$, with terms such that $\lim_{n\to\infty} |z_{n+1}/z_n| = L$:
  - i if $L < 1$ series is absolutely convergent
  - ii if $L > 1$ (including $L = \infty$) series is divergent
  - iii if $L = 1$ is inconclusive
- Root test: series $\sum_{k=1}^{\infty} z_k$, with terms such that $\lim_{n\to\infty} |z_n|^{1/n} = L$:
  - i if $L < 1$ series is absolutely convergent
  - ii if $L > 1$ (including $L = \infty$) series is divergent
  - iii if $L = 1$ is inconclusive

- As presaged by Cauchy's formula's power series are of particular interest

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

- z_0 is the *center of the series*
- Ratio test on $\sum_{k=1}^{\infty} z^{k+1}/k$, absolutely convergent for $|z| < 1$

$$L = \lim_{n\to\infty} \frac{|z^{n+2}/(n+1)|}{|z^{n+1}/n|} = \lim_{n\to\infty} \frac{n}{n+1} |z| = |z|$$

- Circle of convergence: series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$, $L = \lim_{n\to\infty} |a_{n+1}/a_n|$
  i. if $L \neq 0$ series has radius of convergence $R = 1/L$, $|z - z_0| < R$
  ii. if $L = 0$ series converges everywhere $R = \infty$
  iii. if $L = \infty$ radius of convergence is $R = 0$

- $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ represents a continuous function f for $|z-z_0|<R$, $R\neq 0$
- Term-by-term integration is possible for any contour C within $|z-z_0|<R\neq 0$
- Term-by-term differentiation is possible within $|z-z_0|<R\neq 0$
- $f(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k$ is analytic for $|z-z_0|<R$, $R\neq 0$

$$f'(z)=\sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1},\ f''(z)=\sum_{k=2}^{\infty} k(k-1)\, a_k (z-z_0)^{k-2}$$

$$f(z_0)=0!a_0,\ f'(z_0)=1!a_1,\ f''(z_0)=2!\,a_2,\ \ldots$$

- Taylor's theorem $f: D\rightarrow \mathbb{C}$, f analytic in D, $z_0\in D\subset \mathbb{C}$

$$f(z)=\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$