Overview

- Taylor series
- Laurent series
- Singularities, zeros, and poles

- A power series $\sum_{k=0}^{\infty} a_k(z-z_k)^k$ represents a continuous function $f\colon \mathbb{C} \rightarrow \mathbb{C}$ within its radius of convergence $|z-z_0|<R, R\neq 0$
- The power series can be differentiated and integrated term-by-term within $|z-z_0|<R, R\neq 0$
- $f\colon D \rightarrow \mathbb{C}$ analytic, $z_0 \in D$ can be represented by the Taylor series

$$f(z)=\sum_{k=0}^{\infty} a_k(z-z_0)^k=\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

$$f(z_0)=a_0,\ f'(z_0)=a_1,\ f''(z_0)=2a_2,\ \ldots$$

- Algebraic identity

$$\frac{1-t^n}{1-t} = 1 + t + t^2 + \cdots + t^{n-1} \Rightarrow \frac{1}{1-t} = 1 + \cdots + t^{n-1} + \frac{t^n}{1-t}$$

- Cauchy's integral formulas

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,\mathrm{d}z}{(z-z_0)^{n+1}}$$

- $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(z-z_0)^k$
- Start from Cauchy-Goursat

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(s)\,\mathrm{d}s}{s-z} = \frac{1}{2\pi i} \oint \frac{f(s)\,\mathrm{d}s}{s-z_0-(z-z_0)} =$$

$$= \frac{1}{2\pi i} \oint \frac{f(s)\,\mathrm{d}s}{(s-z_0)\left[1-\frac{z-z_0}{s-z_0}\right]}$$

$$= \frac{1}{2\pi i} \oint \frac{f(s)}{(s-z_0)} \left[1 + \frac{z-z_0}{s-z_0} + \cdots + \left(\frac{z-z_0}{s-z_0}\right)^{n-1} + \frac{\left(\frac{z-z_0}{s-z_0}\right)^n}{1-\frac{z-z_0}{s-z_0}}\right] \mathrm{d}s$$

- Recall geometric series, $\frac{1}{1-t} = 1 + \cdots + t^{n-1} + \frac{t^n}{1-t}$, $t = \frac{z-z_0}{s-z_0}$

- After expansion using polynomial factorization identity

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(s)}{(s-z_0)}\left[1 + \frac{z-z_0}{s-z_0} + \dots + \left(\frac{z-z_0}{s-z_0}\right)^{n-1} + \frac{\left(\frac{z-z_0}{s-z_0}\right)^n}{1-\frac{z-z_0}{s-z_0}}\right]\mathrm{d}s \Rightarrow$$

$$2\pi i\, f(z) = \oint \frac{f(s)\,\mathrm{d}s}{(s-z_0)} + (z-z_0)\oint \frac{f(s)\,\mathrm{d}s}{(s-z_0)^2} + \dots + (z-z_0)^{n-1}\oint \frac{f(s)\,\mathrm{d}s}{(s-z_0)^n} + 2\pi i\, R_n$$

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \dots$$

$$R_n = \frac{1}{2\pi i}\oint \frac{f(s)}{s-z_0}\cdot\frac{\left(\frac{z-z_0}{s-z_0}\right)^n}{1-\frac{z-z_0}{s-z_0}}\mathrm{d}s = \frac{1}{2\pi i}\oint \frac{f(s)}{(s-z)}\left(\frac{z-z_0}{s-z_0}\right)^n \mathrm{d}s \to 0, \text{as } n \to \infty$$

- $f\colon D \to \mathbb{C}$ analytic in $D\colon r < |z - z_0| < R$ has the series representation

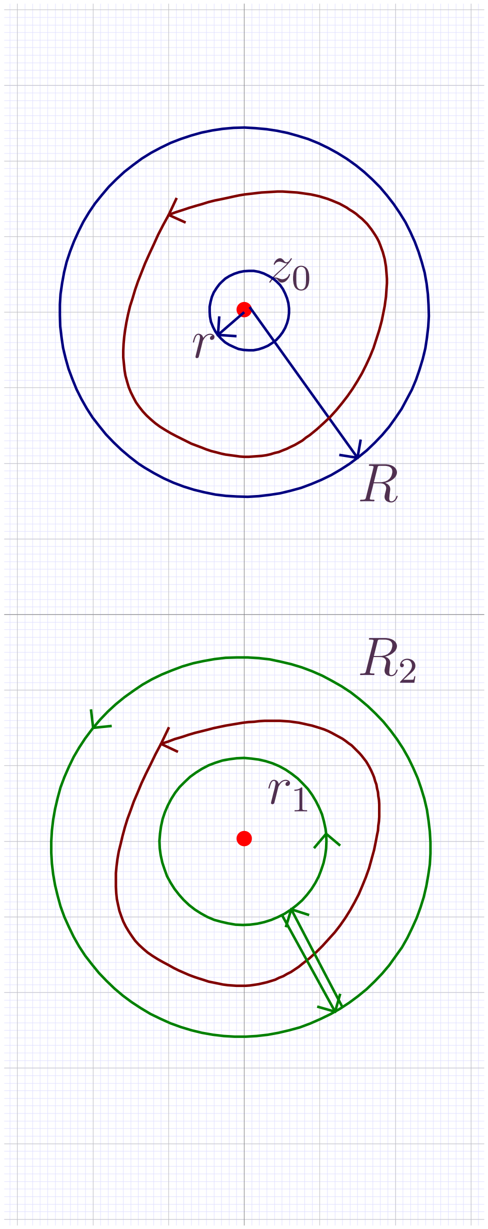
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

where coefficients are defined as

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{k+1}},\ k = 0, \pm 1, \ldots,$$

with C a contour within D.

- Choose r_1, R_2 s.t.: $r < r_1 < R_2 < R$
- $2\pi i\, f(z) = \oint_{C_2} \frac{f(s)\,\mathrm{d}s}{s-z} - \oint_{C_1} \frac{f(s)\,\mathrm{d}s}{s-z}$



1. Integral over C_2 is treated similarly to Taylor series proof

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)\,\mathrm{d}s}{s-z}=\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)\,\mathrm{d}s}{s-z_0-(z-z_0)}=$$

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)\,\mathrm{d}s}{(s-z_0)\left[1-\frac{z-z_0}{s-z_0}\right]}=\sum_{k=0}^{\infty}a_k\,(z-z_0)^k(\text{as in Taylor's theorem})$$

2. Integral over C_1 is analogous, but you factor out a different term

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)\,\mathrm{d}s}{s-z}=\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)\,\mathrm{d}s}{z-z_0-(s-z_0)}=$$

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)\,\mathrm{d}s}{(z-z_0)\left[1-\frac{s-z_0}{z-z_0}\right]}=\sum_{k=1}^{\infty}a_{-k}\,(z-z_0)^{-k}$$

- Points at which $f\colon\mathbb{C}\to\mathbb{C}$ is not analytic are *singularities* of the function, e.g.

$$f_1(z)=\frac{1}{z^2+1}\text{ is singular at }\pm i,\ f_2(z)=\log z\text{ is singular for }\operatorname{Im} z=0, \operatorname{Re} z\leqslant 0$$

- A singularity z_0 is *isolated* if there exists R such that f analytic for

$$0<|z-z_0|<R$$

- A singularity z_0 is *not isolated* if every neighborhood contains another singularity

$$z_0=0\text{ is not an isolated singularity of }f(z)=\log z$$

- A series representation is possible for f with isolated singularities

- $f(z) = 0$, then z is a zero of f
- The function $f(z) = p(z)/q(z)$ has a pole at z if $q(z)$ has a zero at z
- The order of the pole is the number of time z is a repeated root
- Laurent series of functions with k^{th} order poles have terms up to the power z^{-k}
- Define two parts of the Laurent series: $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$
  - *Principal part*: $\sum_{k=-\infty}^{-1} a_k (z - z_0)^k = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k}$
  - Taylor part: $\sum_{k=0}^{\infty} a_k (z - z_0)^k$

- If principal part is zero, z_0 is a *removable singularity*

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

- If principal part has a finite number of terms n, z_0 is a *pole of order n*

$$\frac{1}{(z - z_0)^n}$$

- A pole of order 1 is a *simple pole*

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \dots$$

- If the principal part contains infinitely many terms, z_0 is an *essential singularity*

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!} \cdot \frac{1}{z^3} + \frac{1}{5!} \cdot \frac{1}{z^5} - \dots$$