Overview

- Residue theorem
- Evaluation of real integrals

- Consider $f\colon D\to\mathbb{C}$, f analytic in $D\colon r<|z-z_0|<R$, with Laurent series

$$f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

- Integrate term-by-term on a circle C around z_0, $|z-z_0|<\rho$, $r<\rho<R$

$$\oint_C a_k(z-z_0)^k\,\mathrm{d}z$$

- Recall integrals of monomials on unit circle $z=e^{i\theta}$

$$\oint_{|z|=1}\frac{\mathrm{d}z}{z^2}=i\int_0^{2\pi}e^{-2i\theta}\,e^{i\theta}\,\mathrm{d}\theta=i\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta+\int_0^{2\pi}\sin\theta\,\mathrm{d}\theta=0,$$

$$\oint_{|z|=1}\frac{\mathrm{d}z}{z}=i\int_0^{2\pi}e^{-i\theta}\,e^{i\theta}\,\mathrm{d}\theta=i\int_0^{2\pi}1\cdot\mathrm{d}\theta=2\pi i.$$

- Conclude that the only contribution to

$$\oint_C f(z)\mathrm{d}z = \sum_{k=-\infty}^{\infty} a_k \oint_C (z - z_0)^k \, \mathrm{d}z,$$

comes from the $k = -1$ term

$$\oint_C f(z)\, \mathrm{d}z = 2\pi i\, a_{-1}$$

- Define the *residue* of f at $z = z_0$ a pole of order n by

$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z) \right] = a_{-1}$$

- Let $C = \{z, |z| = 1\}$, i.e., $z = e^{i\theta}, \mathrm{d}z = i\,e^{i\theta}\,\mathrm{d}\theta \Rightarrow \mathrm{d}\theta = (i\,z)^{-1}\,\mathrm{d}z \Rightarrow$

$$I = \oint_C f\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right)\frac{\mathrm{d}z}{i\,z}$$

- Example

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{(2 + \cos\theta)^2} = \oint_C \frac{1}{\left(2 + \frac{1}{2}(z + z^{-1})\right)^2}\frac{\mathrm{d}z}{i\,z} \Rightarrow$$

$$I = \frac{4}{i}\oint_C \frac{z\,\mathrm{d}z}{(z^2 + 4z + 1)^2} = 8\pi\,\mathrm{Res}\,\{f(z), \sqrt{3} - 2\} = \frac{4\pi}{3\sqrt{3}}$$

$$\mathrm{Res}\,\{f(z), \sqrt{3} - 2\} = \lim_{z \to \sqrt{3} - 2} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z - \sqrt{3} + 2)^2 \frac{z}{(z^2 + 4z + 1)^2}\right] = \frac{1}{6\sqrt{3}}$$

- Integrals are scalar products from Fourier series and evaluated as

$$I=\lim_{R\to\infty}\left[\oint_C f(z)\cos(az)\,\mathrm{d}z-\int_S f(z)\cos(az)\,\mathrm{d}z\right]$$

$$J=\lim_{R\to\infty}\left[\oint_C f(z)\sin(az)\,\mathrm{d}z-\int_S f(z)\sin(az)\,\mathrm{d}z\right]$$

on contour formed of line segment along real axis from $-R$ to R and upper semicircle S, $z=Re^{i\theta}, 0\leqslant\theta\leqslant\pi$

- Note that for $z\in\mathbb{C}$, it is possible for $\cos z$, $\sin z>1$
- When integrals over S go to zero as $R\to\infty$, with z_k singularities in $\operatorname{Im} z>0$

$$I=2\pi i\sum_{k=1}^{n}\operatorname{Res}\{f(z)\cos(az),z_k\},\ J=2\pi i\sum_{k=1}^{n}\operatorname{Res}\{f(z)\sin(az),z_k\}$$

- Behavior of integrals over upper half-plane semicircle depends on integrand

Theorem. *(19.6.1)* $f(z) = P(z)/Q(z)$ *a rational function with degrees* n, m *of* P, Q. *When* $m \geqslant n+1$, $\lim_{R\to\infty} \int_S P(z)/Q(z)\,\mathrm{d}z = 0$

Theorem. *(19.6.2)* $f(z) = P(z)/Q(z)$ *a rational function with degrees* n, m *of* P, Q. *When* $m \geqslant n+1$, $\lim_{R\to\infty} \int_S P(z)/Q(z)\, e^{iaz}\,\mathrm{d}z = 0$

- A singularity $z_k \in \mathbb{R}$ must be avoided through a semicircle $z - z_k = r e^{i\theta}$, θ from $\theta_1 = \pi$ to $\theta_2 = 0$

$$\lim_{r\to 0} i r \int_{\pi}^{0} f(z_k + r e^{i\theta})\, e^{i\theta}\,\mathrm{d}\theta = \pi i \,\mathrm{Res}\{f(z), z_k\}$$